



# THE ONSET OF BIFURCATIONS IN THE FORCED DUFFING EQUATION WITH DAMPING

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Received December 28, 2011

The paper presents a complete description of the periodic solutions of the Duffing equation:

$$\ddot{y} + 2\Delta\dot{y} + y^3 = \Gamma \cos t$$

for large values of the forcing  $\Gamma$  and the damping  $\Delta$ . It contains a proof that the equation admits of an infinite sequence of bifurcation curves in the  $\Gamma - \Delta$  plane, alternately of the saddle-node and odd periodic — simply  $2\pi$ -periodic type, whose maxima lie at large  $\Gamma$  along the line:

$$\Delta_c(\Gamma) = \frac{1}{12\pi} \ln \Gamma - \frac{1}{3\pi} \ln \ln \Gamma + C(\Gamma)$$

where  $C(\Gamma)$  has a finite limit as  $\Gamma \rightarrow \infty$ . The positions of the maxima are interlaced in asymptotically equal intervals of  $\Gamma^{1/3}$ , with a spacing of 1.403 units. For  $\Delta > \Delta_c(\Gamma)$ , the Duffing equation admits of a unique periodic solution if  $\Gamma$  is high enough.

These results are obtained by showing that the half-period Poincaré map offered by the Duffing equation is asymptotically equivalent to a map of a circle into itself, according to:

$$\chi \Rightarrow \beta \cos(\chi + \Sigma)$$

where  $\chi$  is an angular variable and  $\beta, \Sigma$  depend on  $\Gamma, \Delta$ . The numerical constants appearing in the circle map and its corrections are determined in the limit  $\Gamma \rightarrow \infty$  by a parameter-free boundary layer equation and its variation.

**Keywords:** Forced duffing equation; bifurcation sequences; bifurcation diagrams; periodic solutions; circle map; Poincaré mapping; asymptotic behavior of differential equations; transition to chaotic motion; averaging methods; WKB approximation; boundary layer theory; singular perturbations.

## 1. Introduction

This paper is concerned with the Duffing equation with external forcing and damping in its simplest form:

$$\ddot{y} + 2\Delta\dot{y} + y^3 = \Gamma \cos \bar{t}. \quad (1)$$

This equation exhibits a large variety of periodic solutions, possibly with periods different from that of the driving force and whose number changes with the values of the parameters  $\Delta, \Gamma$ . These solutions may be studied at low forcing by approximate analytic methods (as in the classical books by Hayashi [1964], Stoker [1950], Hagedorn [1978], Landau and

Lifshitz [1960]) or numerically in larger domains of the parameters  $\Gamma, \Delta$ . There exist well-known diagrams, due to Ueda [1980], in the  $\Gamma - \Delta$  plane of the boundaries of regions where (1) admits a certain type of periodic solutions, e.g. with a given period  $2\pi m/n$ . As the damping is decreased, these plots become increasingly intricate [Ueda, 1980]. For  $\Delta > 0.1$  a diagram similar to Fig. 1 of Ueda [1980] extending to  $\Gamma \approx 200$  may be found in [Höhler, 1993]. (A similar diagram, somewhat less detailed, exists in [Sato *et al.*, 1983].) If, at every fixed  $\Gamma$ ,  $\Delta$  is increased, one reaches always a region where (1) admits a unique  $2\pi$ -periodic solution.

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If a  $2\pi$ -periodic solution  $y_P(\bar{t})$  of (1) is unique, then it is necessarily odd-periodic (i.e. its Fourier series contains only odd harmonics): indeed,  $y_{P1}(\bar{t}) \equiv -y_P(\bar{t} + \pi)$  is also a solution of (1) and, since it is by assumption identical to  $y_P(\bar{t})$ , it follows that, for all  $\bar{t}$ :

$$y_P(\bar{t}) = -y_P(\bar{t} + \pi). \quad (2)$$

From the  $\Gamma - \Delta$  diagrams of [Ueda, 1980] (also of [Höhler, 1993]) one sees that, moving along a line of constant  $\Delta$  (at an intermediate value, say  $0.2 < \Delta < 0.5$ ) starting at  $\Gamma = 0$ , there is first a small  $\Gamma$ -interval where (1) admits a unique (odd-)periodic solution  $y_P(\bar{t})$ ; this solution is also *stable*: it is an attractor in a Poincaré plot of period  $\pi$ . At a certain value of  $\Gamma = \Gamma_{SN}^L$  a *saddle-node* bifurcation occurs and two new solutions, both odd-periodic, appear, one of which is stable, the other unstable. The “earlier” stable solution  $y_P(\bar{t})$  may be continued smoothly past  $\Gamma_{SN}^L$  but “annihilates” at a higher  $\Gamma = \Gamma_{SN}^R$  with the unstable solution originating at  $\Gamma_{SN}^L$  in a *reverse saddle-node* bifurcation. The remaining stable odd periodic solution, now unique, may be continued in  $\Gamma$  up to a point  $\Gamma_F^L$  where it undergoes a *pitchfork* bifurcation of a special type: it becomes unstable when continued past  $\Gamma_F^L$  and two simply  $2\pi$ -periodic (*not* odd-periodic) stable solutions appear for  $\Gamma > \Gamma_F^L$ ; these later disappear again at a higher  $\Gamma_F^R$ , if the damping  $\Delta$  is high enough<sup>1</sup>; at smaller values of  $\Delta$  one traverses first an interval where the two stable solutions above lose their stability and two further  $4\pi$ -periodic solutions appear with a shorter life (in  $\Gamma$ ); if  $\Delta$  is small enough one traverses a whole sequence of bifurcations leading to an attracting chaotic motion, presented in [Ueda, 1980, 1979] in well-known pictures. At values of  $\Gamma$  larger than  $\Gamma_F^R$  one meets again an interval of uniqueness, up to the next saddle-node bifurcation: for  $\Delta$  sufficiently large, the saddle-nodes and flip bifurcations interlace.

A detailed study of Eq. (1) at values of the forcing between ca.850 and ca.1500 and a damping  $\Delta = 0.25$  is the object of a paper by Byatt-Smith [1986]; see also [Byatt-Smith, 1987].<sup>2</sup> The description of bifurcations given above is complicated by the appearance of  $6\pi$ -periodic solutions, which also

generate islands of chaotic motion as one moves up in  $\Gamma$ . There are also windows in  $\Gamma$ , where no chaotic motion exists but, e.g.  $12\pi$ -periodic solutions. A study of (1) by analogue methods up to  $\Gamma \approx 2000$  is presented by Robinson [1989], who points out the periodicity in  $\Gamma^{1/3}$  of the way periodic solutions multiply (and disappear). A recent beautifully illustrated description of the formidable intricacy of the bifurcation diagrams for Eq. (1) is given by Bonatto *et al.* [2008]. Numerical evidence suggests that the sequence of bifurcations e.g. at  $\Delta = 0.2$  [Parlitz & Lauterborn, 1985] or  $\Delta = 0.3$  [Höhler, 1993] is infinite; the positions of the maxima of the saddle-node and flip (pitchfork) bifurcation curves appear to be equidistant in the variable  $\Gamma^{1/3}$  (see also [Sato *et al.*, 1983] for an early attempt to explain this regularity, quite different from the present one).

The question arises whether an analysis of Eq. (1) can explain these phenomena from first principles. Since the complexity of the diagrams increases with decreasing damping  $\Delta$ , it is tempting to start such an attempt from the region of high  $\Delta$  where the solution of (1) is unique and give an explanation for the appearance of bifurcations as the damping becomes smaller. To the knowledge of the author, there exist almost no published descriptions of the domain of uniqueness of the solutions of (1), with the exception of the result of Loud [1955] who shows that, if an harmonic term  $+ky$  is present in (1), then (1) has a unique periodic solution at every fixed  $\Gamma$ , provided  $\Delta$  is large enough (essentially  $\Delta > \text{const} \times \Gamma$ ); it seems, however, that the method is not readily extendable to the situation  $k = 0$  of (1). Since the present work is concerned with the regime of large forcing  $\Gamma$ , I refer to some unpublished internal reports, which show the qualitative behavior of the unique periodic solutions at high damping [Höhler & Stefanescu, 1987] and establish their uniqueness [Stefanescu, 1989] for large enough  $\Gamma$  in a domain above a line  $\Delta(\Gamma)$  for which:

$$\lim_{\Gamma \rightarrow \infty} \frac{\ln \Gamma}{\Delta(\Gamma)} = 0. \quad (3)$$

In a subsequent internal report [Stefanescu, 1990] it was shown that uniqueness is lost as the

<sup>1</sup>The index  $F$  on the values of  $\Gamma$  comes from “flip”: the pitchfork bifurcation of the period  $2\pi$  Poincaré map is in fact a *flip* bifurcation of the half-period Poincaré map (see Secs. 3 and 8).

<sup>2</sup>The asymptotic expansions for the solutions of (1) given in the papers by Byatt-Smith appear also in the present work (Secs. 5 and 6), although derived in a different manner.

damping traverses a line  $\Delta \approx \ln \Gamma$  and that a sequence of bifurcations alternately of saddle-node and pitchfork type develops for lower  $\Delta$  (if  $\Gamma$  is large enough). This result was based on a controlled approximation by means of averaging methods of the (half-period) Poincaré map provided by the Duffing equation (1). The main conclusion was that asymptotically the Poincaré map is well represented by a circle map:

$$\chi \Rightarrow \beta \cos(\chi + \Sigma) \quad (4)$$

with  $\beta, \Sigma$  depending on  $\Gamma$  and  $\Delta$ . The bifurcation structure of (4) is then easy to obtain. This report remained unpublished at that time. Its results were summarized by the present author in an Appendix to the work of Höhler [1993]. The present work is essentially a repetition of the contents of this report, including the correction of some calculational mistakes,<sup>3</sup> a more careful development of the arguments and the addition of some drawings.

It is one of the results of this paper (and of [Stefanescu, 1990]) that the “tips” of the saddle-node and flip bifurcations, which reach up to the highest values of  $\Delta$  at fixed  $\Gamma$ , lie asymptotically along a line (cf. Eq. (252) below):

$$\Delta(\Gamma) = \frac{1}{12\pi} \ln \Gamma - \frac{1}{3\pi} \ln \ln \Gamma + \dots \quad (5)$$

These maxima are predicted to be asymptotically equidistant in the variable  $\Gamma^{1/3}$  with a spacing given in leading order by (cf. Eq. (254) and Fig. 13):

$$\begin{aligned} \delta(\Gamma^{1/3}) &= \Gamma_{F,p}^{1/3} - \Gamma_{SN,p}^{1/3} = \Gamma_{SN,p+1}^{1/3} - \Gamma_{F,p}^{1/3} \\ &\approx 1.403 \approx \frac{\pi}{\sqrt{3} \int_{-\pi/2}^{\pi/2} |\sin t|^{1/3} dt} \end{aligned} \quad (6)$$

The problem of the description of the Poincaré map of (1) at large  $\Gamma$  was taken up again a little later by Eilenberger and Schmidt [1992], Schmidt and Eilenberger [1998]. These authors also derived the circle map (4) as a limit of the Poincaré map using, however, a different approximation scheme. Since these are the only papers which treat (1) in a spirit related to that of the present work — with similar conclusions — I shall sketch in the last section a comparison of the two approaches.

<sup>3</sup>Without consequences for the conclusion.

We introduce next the notation used throughout this work. We change in (1) variables to:

$$x = \frac{y}{\Gamma^{1/3}}, \quad t = \bar{t} - \frac{3\pi}{2}, \quad \varepsilon = \frac{1}{\Gamma^{2/3}}, \quad \mu = \frac{\Delta}{\Gamma^{2/3}} \quad (7)$$

so that it becomes:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + x^3 = \sin t \quad (8)$$

and  $\Gamma \rightarrow \infty$  means  $\varepsilon \rightarrow 0$ , i.e. the coefficient of the second derivative vanishes. The problem of discussing solutions of (8) for small  $\varepsilon$  is a matter of *singular perturbation theory*, as expounded in the books by O'Malley [1974] or Smith [1985]. The question is well known (for linear equations) in the semiclassical treatment of quantum mechanics — the (J)WKB method. Equation (8) is the form of Duffing's equation used throughout this paper.

If one changes the time unit further to  $\hat{t} = t/\sqrt{\varepsilon}$ , (8) becomes:

$$\frac{d^2 x}{d\hat{t}^2} + 2r \frac{dx}{d\hat{t}} + x^3 = \sin(\sqrt{\varepsilon} \hat{t}), \quad r = \frac{\mu}{\sqrt{\varepsilon}} = \frac{\Delta}{\Gamma^{1/3}} \quad (9)$$

This is the limit of extremely slowly varying forcing at small damping if  $\Delta/\Gamma^{1/3}$  vanishes as  $\Gamma \rightarrow \infty$  and is an invitation to apply the adiabatic theorem of classical mechanics [Landau & Lifshitz, 1960; Arnold, 1978]. Equation (9) is the starting point of the approximations to the Poincaré map of (1) developed by Eilenberger and Schmidt [1992].

We give next a summary of the behavior of (8) at high damping; the unique solutions that are obtained are qualitatively different depending on the relative magnitude of  $\Delta$  and  $\Gamma$  (or of  $\mu$  and  $\varepsilon$ ). Assume  $\Delta = \Delta(\Gamma)$  is a monotonically increasing function of  $\Gamma$ , for  $\Gamma \rightarrow \infty$  (large  $\Gamma$ ).

If, as  $\Gamma \rightarrow \infty$ ,  $\mu = \Delta/\Gamma^{2/3} > \mu_0 > 0$ , we change variables in (8) to:

$$z = \mu x, \quad \bar{\varepsilon} = \frac{\varepsilon}{\mu}, \quad \bar{\mu} = \frac{1}{\mu^3} \quad (10)$$

and obtain:

$$\bar{\varepsilon} \ddot{z} + 2\dot{z} + \bar{\mu} z^3 = \sin t. \quad (11)$$

As  $\varepsilon \rightarrow 0$ , (11) reduces to:

$$2\dot{z} + \bar{\mu} z^3 = \sin t. \quad (12)$$



It is easy to show that, if  $\bar{\mu}$  is bounded, (12) admits a unique periodic solution which can be improved by iteration of (11) to a periodic solution  $z_P(t)$  of the latter; further,  $z_P(t)$  is unique [Stefanescu, 1989].

However, if  $\mu \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , Eq. (8) reduces in this limit to:

$$x^3 = \sin t \quad (13)$$

with the solution:

$$x_{00}(t) = (\sin t)^{1/3}. \quad (14)$$

Corrections to  $x_{00}(t)$  cannot be obtained by iterating (8), since the derivatives of  $x_{00}(t)$  at  $t = 0$  are not finite. We expect nevertheless (14) to be a good approximation to periodic solutions of (8) away from  $t = 0$ . The departures of the solutions of (8) from (14) near  $t = 0$  are obtained by a *boundary layer* analysis (the book by Bender and Orszag [1978] contains an excellent introduction to this subject — treated otherwise in detail in the reference manuals on singular perturbation theory [O'Malley, 1974; Smith, 1985]). Let

$$t = \mu^{3/5}\tau, \quad x = \mu^{1/5}\tau \quad (15)$$

so that (8) becomes:

$$\begin{aligned} \frac{\varepsilon}{\mu^{8/5}} \frac{d^2\eta}{d\tau^2} + 2\frac{d\eta}{d\tau} + \eta^3 \\ = \mu^{-3/5} \sin(\tau\mu^{3/5}) = \tau - \frac{\mu^{6/5}}{6}\tau^3 + \dots \end{aligned} \quad (16)$$

To zeroth order in  $\mu^{6/5}$ , we are interested in that solution of (16) which behaves like  $\tau^{1/3}$  as  $\tau \rightarrow \infty$ , so that it *matches*  $x_{00}(t)$ . If  $\varepsilon/\mu^{8/5} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (i.e.  $\Delta/\Gamma^{1/4} \rightarrow \infty$ ), this solution is obtained by improving iteratively the solution of:

$$2\frac{d\eta}{d\tau} + \eta^3 = \tau \quad (17)$$

with the same boundary condition. A “conjunction” of this solution inside the boundary layer with (14) outside it can be improved to a unique (odd-)periodic solution of (8) ([Stefanescu, 1989; Höhler & Stefanescu, 1987]; see Sec. 3 of this work for related procedures).

If, however,  $\Delta/\Gamma^{1/4} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , the appropriate boundary layer quantities are:

$$t = \varepsilon^{3/8}\tau, \quad x = \varepsilon^{1/8}\eta, \quad \gamma = \frac{\mu}{\varepsilon^{5/8}} \left( = \frac{\Delta}{\Gamma^{1/4}} \right) \quad (18)$$

in terms of which (8) becomes:

$$\begin{aligned} \frac{d^2\eta}{d\tau^2} + 2\gamma\frac{d\eta}{d\tau} + \eta^3 &= \varepsilon^{-3/8} \sin(\varepsilon^{3/8}\tau) \\ &= \tau - \varepsilon^{3/4}\frac{\tau^3}{6} + \dots \end{aligned} \quad (19)$$

As  $\varepsilon \rightarrow 0$ , the solutions of (19) obeying  $\eta(\tau) \approx \tau^{1/3}$  as  $\tau \rightarrow -\infty$  are oscillatory as  $\tau \rightarrow +\infty$  and are damped in a “time”  $\tau \approx 1/\gamma \rightarrow \infty$ .

We distinguish thus three regimes of (8) for large  $\Gamma$ : (i)  $\mu > \mu_0 > 0$  as  $\varepsilon \rightarrow 0$ ; (ii)  $\mu < \mu_0$  and  $\varepsilon/\mu^{8/5} < \text{const}$  as  $\varepsilon \rightarrow 0$ ; (iii)  $\mu \rightarrow 0$ ,  $\gamma = \mu/\varepsilon^{5/8} < \gamma_0$  as  $\varepsilon \rightarrow 0$ . Situations (i), (ii) lead to unique periodic solutions of (8) for large  $\Gamma$ ; for a proof see the internal report [Stefanescu, 1989]. The transition to nonuniqueness occurs in region (iii) (cf. Eqs. (3) and (5)). We shall thus assume throughout the present work that:

$$\gamma = \frac{\Delta}{\Gamma^{1/4}} = \frac{\mu}{\varepsilon^{5/8}} < \gamma_0 \quad \text{as } \varepsilon \rightarrow 0 \quad (20)$$

and use the notation of (18). In view of (5), we find it convenient to use instead of  $\mu$  the variable:

$$\kappa = \frac{\mu}{\varepsilon \ln\left(\frac{1}{\varepsilon}\right)} = \frac{3}{2} \frac{\Delta}{\ln \Gamma} \quad (21)$$

so that bifurcations occur when  $\kappa = O(1)$  as  $\varepsilon \rightarrow 0$ .

The paper is organized as follows: in Sec. 2, some general preparatory statements are established concerning the boundedness of the solutions of (8) and the manner in which they approach each other in time. Section 3 introduces the *inner* and *outer* expansions associated to (8); these are combined and improved to two special, nonoscillatory solutions  $X_L(t), X_R(t)$  of (8), defined for  $t < 0(L)$ ,  $t > 0(R)$  in turn. These solutions<sup>4</sup> are taken as references for  $t < 0$ ,  $t > 0$  in turn and the Poincaré map  $\mathbb{P}$  is defined in terms of the differences:

$$\begin{aligned} v_L(t) &= x(t) - X_L(t), \quad t < 0; \\ v_R(t) &= x(t) - X_R(t), \quad t > 0. \end{aligned} \quad (22)$$

<sup>4</sup>They are called *creeping* solutions by Eilenberger and Schmidt [1992].



In Sec. 4 a precise bound is derived on the region  $\mathbb{D}_0(\varepsilon)$  of phase space where invariant sets of the map  $\mathbb{P}$  may exist. Sections 5 and 6 establish controlled approximations for *quarter period* Poincaré maps  $\mathbb{P}_L, \mathbb{P}_R$  relating sections at  $t = -\pi/2$  and  $t = 0$  ( $\mathbb{P}_L$ ) and  $t = 0$  and  $t = \pi/2$  ( $\mathbb{P}_R$ ) in turn. In Sec. 7 the complete (*half-period*) Poincaré map  $\mathbb{P}(\varepsilon, \kappa)$  is written down and its limiting form for  $\varepsilon \rightarrow 0$ , the circle map  $\Pi$  (4), is established. Finally Sec. 8 discusses the extent to which bifurcation properties of the circle map can be transferred to those of the complete mapping  $\mathbb{P}(\varepsilon, \kappa)$  for small, nonvanishing  $\varepsilon$ . In particular, the statements of the Abstract concerning the asymptotic distribution of the bifurcation lines are derived. The paper is closed with some general remarks and a short comparison with related papers of Eilenberger and Schmidt [1992, 1998].

## 2. General Preparation

### 2.1. Eventual boundedness of motions

**Lemma 2.1.** *There exists a rectangle:*

$$\mathbb{R} : |x(t)| < B_1, \quad \left| \frac{dx}{dt} \right| < \frac{B_2}{\sqrt{\varepsilon}} \quad (23)$$

*so that all solution paths  $(x(t), \dot{x}(t))$  of (8) eventually get inside it. The constants  $B_1, B_2$  are independent of  $\varepsilon, \mu$  if  $\mu$  and  $\varepsilon/\mu$  are sufficiently small.*

*Proof.* The argument is inspired by and similar to the one due to Yoshizawa [1953c, 1953b, 1953a] and presented in the book by Sansone and Conti [1964]. We consider the Lyapunov-type function  $L(p, x, t)$  given by: ( $p = dx/dt$ )

$$L(p, x, t) = E(p, x, t) + D(p, x) \quad (24)$$

$$E(p, x, t) = \varepsilon \frac{p^2}{2} + \frac{x^4}{4} - x \sin t \quad (25)$$

$$\begin{aligned} D(p, x) &= 0 && \text{if } p > \max \left[ \left( \frac{|x|}{\mu} \right)^{1/2}, \left( \frac{A}{\mu} \right)^{1/2} \right] \\ &= \varepsilon \left( p - \left( \frac{x}{\mu} \right)^{1/2} \right) && \text{if } |p| < \left( \frac{x}{\mu} \right)^{1/2}, \quad x > A \\ &= -2\varepsilon \left( \frac{x}{\mu} \right)^{1/2} && \text{if } p < -\left( \frac{x}{\mu} \right)^{1/2}, \quad x > A \\ &= -2\varepsilon \frac{x}{A} \left( \frac{|x|}{\mu} \right)^{1/2} && \text{if } p < -\left( \frac{A}{\mu} \right)^{1/2}, \quad |x| < A \\ &= 2\varepsilon \left( \frac{|x|}{\mu} \right)^{1/2} && \text{if } p < -\left( \frac{|x|}{\mu} \right)^{1/2}, \quad x < -A \\ &= -\varepsilon \left( p - \left( \frac{|x|}{\mu} \right)^{1/2} \right) && \text{if } |p| < \left( \frac{|x|}{\mu} \right)^{1/2}, \quad x < -A \end{aligned} \quad (26)$$

where  $A$  is a constant, which will be chosen appropriately in the following. The function  $L(p, x, t)$  is continuous and piecewise differentiable. In opposition to the functions considered in [Sansone & Conti, 1964] the function  $L$  in (24) is time-dependent, however in a “mild” manner: it is

$2\pi$ -periodic. The choice of  $D(p, x)$  is a modification of the one used by Reuter [1951], also presented<sup>5</sup> in [Sansone & Conti, 1964, p. 376, Chapter VII, Section 3]. Differentiation of (24) and use of the Duffing equation (8) establishes that:

<sup>5</sup>Use of the function offered for a more general situation by Reuter turns out to be appropriate only if the damping  $\Delta$  increases faster than  $\Gamma^{1/3}$ .

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$$\begin{aligned} \frac{dL}{dt} &= -2\mu p^2 - x \cos t + \frac{\partial D}{\partial x} p \\ &\quad - \frac{1}{\varepsilon} \frac{\partial D}{\partial p} (2\mu p + x^3 - \sin t) \\ &< -\delta < 0, \quad \text{if } x, p \in \mathbf{C}R_0, \\ R_0 &= \left\{ |x| < A, |p| < \left( \frac{A}{\mu} \right)^{1/2} \right\} \end{aligned} \quad (27)$$

if, e.g.  $A \leq 2$  and  $\mu, \varepsilon$  are appropriately small (depending on  $A$ , e.g. for  $A = 2$ ,  $\mu < 1/2$ ,  $\varepsilon/\mu < 1/2$ ). The quantity  $\delta$  is independent of  $(x, p)$  in  $\mathbf{C}R_0$ . As a consequence of (27), for any trajectory  $(x(t), p(t))$  which starts at  $t = t_0$  in  $\mathbf{C}R_0$  and for any finite interval  $\Delta t$ ,

$$\begin{aligned} L(x(t + \Delta t), p(t + \Delta t), t + \Delta t) - L(x(t), p(t), t) \\ < \text{const} < 0 \end{aligned} \quad (28)$$

as long as the trajectory stays in  $\mathbf{C}R_0$ .

For any  $(x, y)$  outside  $R_0$  we define two functions, related to (24):

$$\begin{aligned} L_M(x, p) &= \varepsilon \frac{p^2}{2} + \frac{x^4}{4} + D(x, p) + |x| \\ L_m(x, p) &= \varepsilon \frac{p^2}{2} + \frac{x^4}{4} + D(x, p) - |x| \end{aligned} \quad (29)$$

which both increase indefinitely as  $|x|, |p|$  tend to  $\infty$  along any direction in the  $(x, p)$  plane, uniformly with respect to the direction. Further, we

can replace the rectangle  $R_0$  with a larger one  $R_1$ , so that both  $L_M, L_m$  are monotonically increasing outside  $R_1$  along any ray in the  $(x, p)$  plane. One verifies that e.g. the rectangle

$$R_1 : \left\{ |x| < A, |p| < \frac{1}{\sqrt{\varepsilon}} \right\} \quad (30)$$

fulfills this condition. The various domains of the  $(x, \sqrt{\varepsilon}p)$  plane that appear in the definition of  $D(p, x)$ , Eq. (26) are shown in Fig. 1 for the situation  $\varepsilon/\mu = 1/2$ ,  $\varepsilon = 1/4$ . The  $\varepsilon$ -dependence becomes weaker as  $\varepsilon \rightarrow 0$  at constant  $\varepsilon/\mu$  with this choice of variables, since the changes of  $D(x, \sqrt{\varepsilon}p)$  between the various domains are proportional to  $\sqrt{\varepsilon}$ . The rectangle  $R_0$  outside which  $dL/dt < 0$  is shown with a dotted line, the “increased” rectangle  $R_1$ , Eq. (30) with a continuous line. Let now

$$L_{\lim} \equiv \max_{(x,p) \in \partial R_1} L_M(x, p) \quad (31)$$

where  $\partial R_1$  denotes the boundary of  $R_1$ . By our choice of  $R_1$ , for a trajectory starting in  $\mathbf{C}R_1$ , as long as the corresponding function  $L(x(t), p(t), t)$  assumes values strictly larger than  $L_{\lim}$ , it is certain not to leave  $\mathbf{C}R_1$ . Indeed, by (31) crossing  $\partial R_1$  requires  $L \leq L_{\lim}$ . But in  $\mathbf{C}R_1$ , the value  $L(x(t), p(t), t)$  decreases monotonically with time by (28) so that, for any value  $L_1 > L_{\lim}$  there exists a time  $t$  at which  $L(x(t), p(t), t) = L_1$ .

Consider now such a value  $L_1$  and the closed region  $\bar{\mathcal{R}}$  in the  $(x, p)$  plane delimited by the closed

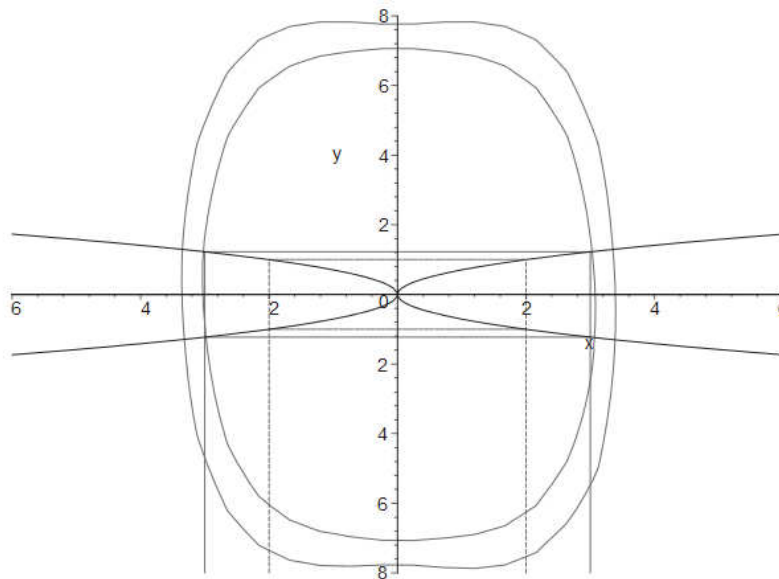


Fig. 1. The rectangles  $R_0, R_1$  used for  $D(p, x)$  and the curves  $\mathcal{C}_{\lim}, \mathcal{C}_m$ .

curve:

$$C_m : L_m(x, p) = L_1. \quad (32)$$

It contains the rectangle  $R_1$  strictly in its interior. According to the above, all trajectories starting in  $CR_1$  reach at some time  $t$  the region  $\bar{\mathcal{R}}$  since their corresponding  $L$ -function achieves the value  $L_1$ . Such a trajectory cannot leave the domain  $\bar{\mathcal{R}}$  by traversing (or turning back from) the curve  $C_m$  because outside  $C_m$ ,  $L > L_m(x, p) > L_1$  and this would contradict (28). Thus the trajectory is “trapped” in  $\bar{\mathcal{R}}$ . We obtain the statement of Lemma 2.1 by choosing  $\mathbb{R}$  as a rectangle containing  $C_m$  in its interior: one verifies that this is so if  $B_1 \approx 4$ ,  $B_2 \approx 8$ . This ends the proof of Lemma 2.1.

The curves  $C_{\text{lim}}$ , corresponding to  $L_M(x, p) = L_{\text{lim}}$ , Eq. (31),  $C_m$  of (32) (enclosing  $C_{\text{lim}}$ ) are shown in Fig. 1.

## 2.2. The approach to some special solutions

The following describes the manner in which a solution  $x(t)$  of (8), trapped inside the rectangle  $\mathbb{R}$  of Lemma 2.1, approaches a solution  $x_0(t)$  also contained in  $\mathbb{R}$  and subjected to the following supplementary

**Condition 2.1.** *There exist  $a, b > 0$ , independent of  $\varepsilon$ , so that, for all  $t$  in some interval  $[t_1, t_2]$  with  $0 < t_1 < t_2 < \pi \pmod{\pi}$*

$$|x_0(t, \varepsilon)| > a, \quad \left| \frac{dx_0}{dt}(t, \varepsilon) \right| < b. \quad (33)$$

Solutions of (8) contained in the rectangle  $\mathbb{R}$  and obeying this condition will be shown to exist in Sec. 3. For any other  $x(t)$  staying in  $\mathbb{R}$  for  $t > t_0$  we may state:

**Lemma 2.2.** *Assume  $\mu/\varepsilon^{1/2} = \Delta/\Gamma^{1/3} < A_0$  and  $\varepsilon/\mu = 1/\Delta \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then, for  $\varepsilon$  sufficiently small, there exist constants  $K, C$ , independent of  $\varepsilon$ , so that, for any solution confined to the rectangle  $\mathbb{R}$  of (23) and for  $t \in [t_1, t_2]$ :*

$$\begin{aligned} & \max \left[ |x(t) - x_0(t)|, \varepsilon^{1/2} \left| \frac{dx}{dt}(t) - \frac{dx_0}{dt}(t) \right| \right] \\ & < K e^{-C\mu(t-t_1)/\varepsilon}. \end{aligned} \quad (34)$$

*Proof.* Let  $C < 1$  and

$$u(t) = (x(t) - x_0(t))e^{C\mu(t-t_1)/\varepsilon}. \quad (35)$$

It verifies:

$$\begin{aligned} & \varepsilon \frac{d^2 u}{dt^2} + 2\mu(1-C) \frac{du}{dt} \\ & + u \left[ 3x_0(t)^2 + \frac{\mu^2(C^2 - 2C)}{\varepsilon} \right] \\ & + 3u^2 x_0(t) e^{-C\mu(t-t_1)/\varepsilon} \\ & + u^3 e^{-2C\mu(t-t_1)/\varepsilon} = 0. \end{aligned} \quad (36)$$

Consider the Lyapunov function:

$$L_u = \frac{1}{2} \left( \varepsilon \frac{du}{dt} + 2\mu \bar{C} u \right)^2 + \varepsilon G(u, t) \quad (37)$$

with  $\bar{C} \equiv 1 - C$  and

$$G(u, t) = \int_0^u u' F(u', t) du' \quad (38)$$

where  $uF(u, t)$  denotes the last three terms of the l.h.s. of (36). The forms  $F(u, t)$ ,  $G(u, t)$  are positive definite for  $t \in [t_1, t_2]$  if, e.g.  $C < \min(1/2, a^2/(4A_0^2))$ . Using (36) we get:

$$\begin{aligned} \frac{dL_u}{dt} &= -2\mu \bar{C} \left( u^2 F(u, t) - \frac{\varepsilon}{2\mu \bar{C}} \frac{\partial G}{\partial t} \right) \\ &\equiv -2\mu \bar{C} u^2 H(u, t). \end{aligned} \quad (39)$$

One verifies that, with the choice of  $C$  above, if  $\varepsilon/\mu = 1/\Delta$  is sufficiently small,  $H(u, t)$  (of (39)) is positive definite for  $t \in [t_1, t_2]$ . Thus the solution paths  $(u, du/dt)$  stay contained in the bounded domains defined at every  $t \in [t_1, t_2]$  by:

$$L_u(t) < L_u(t_1). \quad (40)$$

But  $L_u(t_1) = O(\varepsilon)$ , since  $(u, du/dt)$  are contained in the rectangle  $\mathbb{R}$  of Lemma 2.1 and  $\mu/\sqrt{\varepsilon} < A_0$ . It follows from (40) and (37) that  $G(u, t) = O(1)$  for  $t \in [t_1, t_2]$  and since  $G(u, t) = u^2 F_1(u, t)$  with  $F_1(u, t)$  strictly positive definite, it follows that  $u(t) = O(1)$  for  $t \in [t_1, t_2]$ . Further, since  $\varepsilon du/dt + 2\mu \bar{C} u = O(\varepsilon^{1/2})$ , and  $\mu/\sqrt{\varepsilon}$  is bounded, we conclude that  $du/dt = O(\varepsilon^{-1/2})$  for  $t \in [t_1, t_2]$ . Returning now to (35) we obtain (using again the bound on  $\mu/\sqrt{\varepsilon}$  for the evaluation of the time derivatives) the statement of the lemma.

The bound  $\mu/\sqrt{\varepsilon} < A_0$  delimits in the  $\Gamma - \Delta$  plane a region where the damping may still be quite



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large to ensure at small  $\varepsilon$ , uniqueness of the periodic solution of (8). In this paper the concern is the region of relatively small damping ( $\Delta \approx \text{const} \ln \Gamma$ ) where uniqueness gets lost. The domain of larger damping may be treated completely, as is shown in the unpublished report [Stefanescu, 1989].

### 3. Inner and Outer Expansions. Reference Solutions

#### 3.1. Left-hand reference solution

##### 3.1.1. The outer expansion

For small  $\varepsilon$  and  $\mu$  and for  $t$  away from  $n\pi$ , we expect to obtain an approximate solution  $x(t)$  of

$$\varepsilon \ddot{x} + 2\mu \dot{x} + x^3 = \sin t \quad (41)$$

by simply starting with:

$$x_{00}(t) = (\sin t)^{1/3} \quad (42)$$

and determining step by step with the help of (41) the coefficients  $x_{kl}(t)$  of an expansion:

$$x_o(t) \equiv \sum_{k,l} \mu^k \varepsilon^l x_{kl}(t). \quad (43)$$

Equating to zero the coefficients of the various powers of  $\mu$  and  $\varepsilon$  we obtain successively:

$$x_{10}(t) = -\frac{2\ddot{x}_{00}}{3x_{00}^2}, \quad x_{01}(t) = -\frac{\ddot{x}_{00}}{3x_{00}^2}, \dots \quad (44)$$

and so on, so that we may state, in general:

**Lemma 3.1.** *With the definition in (43):*

$$x_{kl}(t) = t^{1/3-5k/3-8l/3} \sum_q a_{klq} t^{2q} \quad (45)$$

where the sum is uniformly and absolutely convergent for  $t$  in  $[-\pi + \sigma, \pi - \sigma]$ , for any  $\sigma > 0$ .

The proof is done by induction: the set of coefficients  $\{x_{k0}(t)\}$  and  $\{x_{0l}(t)\}$  form “closed” groups allowing the recursive determination of  $x_{k0}(t)$  in terms of  $x_{k'0}(t)$  with  $k' < k$  and  $x_{0l}(t)$  in terms of  $x_{0l'}$  with  $l' < l$ . The coefficient  $x_{kl}(t)$  may be determined in terms of  $x_{k'0l'}$  with  $k' < k, l' \leq l$  and  $k' \leq k, l' < l$ . Thus, we may determine successively the sets  $\{x_{1l}\}$ ,  $\{x_{2l}\}$ , etc. From (44), we see that,

e.g.

$$x_{10}(t) = -\frac{2}{9} t^{-4/3} \cos t \left( \frac{t}{\sin t} \right)^{4/3},$$

$$x_{01}(t) = \frac{2}{27} \frac{2 + \sin^2 t}{t^{4/3}} \left( \frac{t}{\sin t} \right)^{7/3}.$$

The ratio  $t/\sin t$  is a holomorphic, even and zero-free function of  $t$  in a disk of radius  $\pi - \sigma$  around the origin; the same is true of  $(t/\sin t)^{4/3}$  which justifies the statement about the convergence of the series in (45) in this situation. The same is true for  $x_{01}(t)$  and is then transmitted recursively to all other coefficients. This ends the justification of Lemma 3.1.

The coefficients  $a_{klq}$  in (45) may be obtained step by step directly as follows: let  $\alpha = \mu/t^{5/3}$ ,  $\beta = \varepsilon/t^{8/3}$  and denote by:

$$\mathcal{D}(x) \equiv \varepsilon \ddot{x} + 2\mu \dot{x} + x^3. \quad (46)$$

Then one verifies that Eq. (41) means (vgl. (45)):

$$\mathcal{D} \left( t^{1/3} \sum_{k,l,q} a_{klq} \alpha^k \beta^l t^{2q} \right) = \sum_{k,l,q} b_{klq} \alpha^k \beta^l t^{2q+1} = \sum_q (-1)^q \frac{t^{2q+1}}{(2q+1)!}, \quad (47)$$

where  $b_{klq}$  are combinations of  $a_{k'0l'q'}$  with  $k' \leq k, l' \leq l$  and  $q' \leq q$ , but with only one term containing  $a_{klq}$  namely  $3a_{000}^2 a_{klq}$ . This equation allows the recurrent determination of  $a_{klq}$  by equating the coefficients of  $\alpha^k \beta^l t^{2q+1}$  on both sides, starting with  $a_{000} = 1$ . The coefficients with  $q = 0$  (or  $k = 0$  or  $l = 0$ ) build a “closed” group: the  $a_{kl0}$  may be determined successively from  $a_{000}$ . Then the calculation of  $a_{klq}$  for  $q > 0$  requires  $a_{k'0l'q'}$  with at least one strict inequality in the set ( $k' \leq k, l' \leq l, q' \leq q$ ).

An obvious question is: to what extent do we satisfy (41) if we restrict ourselves to a  $(K, L)$  truncation of (43)? From (47) we see that terms containing  $b_{klq}$  with  $k > K$  or  $l > L$  are, in general, nonvanishing, so that the action of  $\mathcal{D}$  produces terms of  $O(t\alpha^{K+1}, t\beta^{L+1})$ , i.e.  $O(t\mu^{(K+1)}/t^{5(K+1)/3} + t\varepsilon^{(L+1)}/t^{8(L+1)/3})$ . This shows that a truncation of (43) is approximately a solution only for  $t$  away from zero, i.e. outside the boundary layer, which justifies the label *outer expansion*.

### 3.1.2. The inner expansion

The substitutions:  $x = \varepsilon^{1/8}\eta$ ,  $t = \varepsilon^{3/8}\tau$  change (41) to:

$$\frac{d^2\eta}{d\tau^2} + 2\gamma\frac{d\eta}{d\tau} + \eta^3 = \varepsilon^{-3/8} \sin(\varepsilon^{3/8}\tau) \quad (48)$$

where  $\gamma = \mu/\varepsilon^{5/8}$ . It is natural to look for a solution of (48) as an “inner expansion” in terms of the parameter  $\varepsilon^{3/4}$ :

$$\eta(\tau) = \eta_0(\tau) + \varepsilon^{3/4}\eta_1(\tau) + \varepsilon^{3/2}\eta_2(\tau) + \dots \quad (49)$$

where the  $\eta_q(\tau)$  are in turn solutions behaving like  $\tau^{2q+1/3}$  as  $\tau \rightarrow -\infty$  of the differential equations:

$$\frac{d^2\eta_0}{d\tau^2} + 2\gamma\frac{d\eta_0}{d\tau} + \eta_0(\tau)^3 = \tau \quad (50a)$$

$$\frac{d^2\eta_1}{d\tau^2} + 2\gamma\frac{d\eta_1}{d\tau} + 3\eta_0(\tau)^2\eta_1(\tau) = -\frac{\tau^3}{6} \quad (50b)$$

$$\begin{aligned} \frac{d^2\eta_2}{d\tau^2} + 2\gamma\frac{d\eta_2}{d\tau} + 3\eta_0(\tau)^2\eta_2(\tau) \\ + 3\eta_1(\tau)^2\eta_0(\tau) = \frac{\tau^5}{5!}, \text{ etc.} \end{aligned} \quad (50c)$$

Because we wish the solution  $\eta(\tau)$  to approach at large  $-\tau$  the function  $(\sin t)^{1/3}/\varepsilon^{1/8}$ , it is natural to choose those solutions of (50a), (50b), etc. which behave like  $\tau^{2q+1/3}$  as  $-\tau \rightarrow \infty$ , corresponding to the terms  $(-1)^q \varepsilon^{3q/4} \tau^{2q+1/3}/(2q+1)!$  of the Taylor expansion. Concerning the expansion (49) it is easy to show:

**Lemma 3.2.** *The solutions  $\eta_q$  appearing in Eq. (49) exist and are uniquely determined by the requirement  $\eta_q \approx \tau^{2q+1/3}$  as  $\tau \rightarrow -\infty$ . Their asymptotic expansion for  $\tau \rightarrow -\infty$  is given by:*

$$\eta_q(\tau) = \tau^{2q+1/3} \sum_{k,l} a_{klq} \gamma^k \tau^{-5k/3-8l/3}, \quad q = 0, 1, 2, \dots \quad (51)$$

with the same  $a_{klq}$  as in (45).

As before, one verifies by induction that (51) with coefficients  $\tilde{a}_{klq}$  is a consistent asymptotic approximation for the solutions Eqs. (50a), (50b), etc. obeying  $\eta_q(\tau) \approx \tau^{2q+1/3}$  as  $\tau \rightarrow -\infty$ . The coefficients  $\tilde{a}_{klq}$  are determined by substituting the asymptotic

series (51) for each  $\eta_q(\tau)$  in Eq. (49) and requiring that (48) be verified to all orders in  $\tau$  for all  $\varepsilon$  and  $\gamma$ . Denoting  $t' \equiv \varepsilon^{3/8}\tau$ ,  $\alpha' \equiv \gamma\tau^{-5/3}$ ,  $\beta' \equiv \tau^{-8/3}$  and by  $\tilde{\mathcal{D}}$  the action of the right-hand side of Eq. (48) on functions  $\eta(\tau)$ , one verifies:

$$\begin{aligned} \tilde{\mathcal{D}} \left( \sum_{klq} \tau^{1/3} \varepsilon^{3q/4} \tilde{a}_{klq} \gamma^k \tau^{-5k/3-8l/3+2q} \right) \\ = \tau \left( \sum_{klq} \tilde{b}_{klq} (t')^{2q} (\alpha')^k (\beta')^l \right) \\ = \tau \sum \frac{(-1)^q (t')^{2q}}{(2q+1)!} \end{aligned} \quad (52)$$

with  $\tilde{b}_{klq}$  analogous to  $b_{klq}$  of Eq. (47). Equation (52) allows a recurrent determination of  $\tilde{a}_{klq}$  starting from  $\tilde{a}_{000} = 1$ . From the equations determined from (52)  $\tilde{a}_{klq}$  are identical to the corresponding ones in (47) for  $a_{klq}$  if we replace  $t' \leftrightarrow t$ ,  $\alpha' \leftrightarrow \alpha$ ,  $\beta' \leftrightarrow \beta$ . This shows that, indeed,  $\tilde{a}_{klq} \equiv a_{klq}$ .

We notice that the calculation of the asymptotic behavior of  $\eta_q$ ,  $q > 0$  requires knowledge of the behaviors of  $\eta_{q'}$  for  $q' < q$ , since  $\eta_{q'}$  occurs in the equation for  $\eta_q$  (vgl. Eq. (50c)). The relation between outer and inner expansions is apparent if we substitute  $\tau = t\varepsilon^{-3/8}$ ,  $\eta = x\varepsilon^{-1/8}$  in (45): one obtains the sum over  $q$  of the series (51), i.e. the asymptotic behavior (at large  $-\tau$ ) of the inner expansion is the same as the low  $t$  behavior of the outer expansion.

We now show that, indeed, the requirement concerning the asymptotic behavior selects unique solutions of (50a)–(50c). We shall show directly that unique solutions exist admitting (51) as asymptotic expansion, from which the previous statement will follow. Using the method of the variation of parameters we set up an integral equation for the difference

$$u_q(\tau) \equiv \eta_q(\tau) - \eta_q(K, L) \quad (53)$$

where  $\eta_q^{(K,L)}$  is the truncation of (51) after  $(K, L)$  terms. For  $q = 0$  the equation reads:

$$\begin{aligned} u_0(\tau) = \int_{-\infty}^{\tau} e^{-\gamma(\tau-\tau')} (v_1(\tau)v_2(\tau') - v_2(\tau)v_1(\tau')) \\ \times (H(\tau') - 3\eta_0^{(K,L)}u(\tau')^2 - u(\tau')^3) d\tau' \end{aligned} \quad (54)$$



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where

$$H(\tau) \equiv -\left(\frac{d^2\eta_0^{(K,L)}}{d\tau^2} + 2\gamma\frac{d\eta_0^{(K,L)}}{d\tau} + (\eta_0^{(K,L)})^3 - \tau\right) \\ = \gamma^K \tau^{-5K/3-8L/3} O(\gamma\tau^{-2/3}, \tau^{-5/3}) \quad (55)$$

and

$$v_{1,2}(\tau)e^{-\gamma\tau} \approx \frac{e^{-\gamma\tau}}{3^{1/4}|\tau|^{1/6}} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} \left( \frac{3\sqrt{3}}{4} \tau^{4/3} \right) \quad (56)$$

are two linearly independent solutions of the equation:

$$\frac{d^2v}{d\tau^2} + 2\gamma\frac{dv}{d\tau} + 3(\eta_0^{(K,L)})^2v(\tau) = 0.$$

The approximation indicated in Eq. (56) is related to the WKB approximation and is discussed in a related context in Appendix C. One can show now by well-known methods that Eq. (54) admits a unique solution in a space of bounded continuous functions on  $(-\infty, -a)$ ,  $a > 0$  with the norm  $\sup |\tau^{5K/3+8L/3+1}u(\tau)|$ . This solution is obtained by iteration of (54) which also sets a bound on the error made by truncating (51) at the  $(K, L)$  step: it is  $O(\gamma^K \tau^{-5K/3-8L/3-1})$ . Since the order of magnitude of the error is smaller than the last term included, this shows that the expansion (51) is indeed asymptotic; for  $K, L = 0$  we obtain the statement of Lemma 3.2 for  $q = 0$ .

For  $q > 0$ , Eqs. (50a), (50b) and their analogues are linear and so are the corresponding integral equations (54); the solutions are given simply by the integral over (correspondingly modified) terms like  $H(\tau)$  of (55); these terms contain solutions  $\eta_{q^0}$  with  $q' < q$ . The error of truncation after step  $(K, L)$  is now  $C(q)\tau^{2q-1-5K/3-8L/3}\gamma^K$ , with  $C(q)$  a  $(q$ -dependent) constant.

### 3.1.3. The complete left-hand reference solution

Although it is intuitively clear that Eqs. (45) and (49) (using the expansion (51)) are expansions of the same solution of Eq. (41), it is not true that truncations at increasing  $K, L$  fulfill (41) increasingly well on the whole interval  $[-\pi/2, 0]$ . Following methods related to those of O'Malley [1974, Chapter IV] and Smith [1985, Chapter VI], we show how

the two expansions have to be combined to yield a uniform approximation of the solution on  $[-\pi/2, 0]$ . We denote by:

$$x_o^{(K,L)} \equiv \sum_{k \leq K, l \leq L} \mu^k \varepsilon^l x_{kl}(t), \\ x_i^{(Q)} \equiv \varepsilon^{1/8} \sum_{q \leq Q} \varepsilon^{3q/4} \eta_q(\tau) \quad (57)$$

i.e. the  $(K, L)$ -,  $Q$ -truncations of the sums in (45), (49) and take a number  $d$ ,  $0 < d < 3/8$ . With these we set up a  $(K, L, Q)$ -approximant to a solution of (41):

$$x_a(t) \equiv \chi_o(t, \varepsilon^d) x_o^{(K,L)}(t) + \chi_i(t, \varepsilon^d) x_i^{(Q)}(t), \quad (58)$$

where  $\chi_o(t, \varepsilon^d)$  is of class  $\mathbb{C}^2$ ,  $= 0$  for  $t > -a\varepsilon^d$  and  $= 1$  on  $[-\pi/2, -b\varepsilon^d]$ ,  $0 < a < b$ ; further for  $-\pi/2 < t < 0$ ,

$$\chi_i(t, \varepsilon^d) = 1 - \chi_o(t, \varepsilon^d).$$

The function  $x_a(t)$  is not a solution of (41) but is uniformly close for small  $\mu$  and  $\varepsilon$  to such a solution on  $-\pi/2 < t < 0$ . Indeed, substitution of  $x_o^{(K,L)}$  of (54) in (41) leaves terms of  $O((\varepsilon/t^{8/3})^{L+1} + (\mu/t^{5/3})^{K+1})$  uncompensated;  $x_i^{(Q)}$  verifies (41) up to terms of  $O(\varepsilon^{3(Q+1)/4+3/8}\tau^{2(Q+3)})$ ; thus, using the notation of (46), for  $-\pi/2 < t < -a\varepsilon^d$  and  $-b\varepsilon^d < t < 0$  in turn:

$$|\mathcal{D}(x_o^{(K,L)}) - \sin t| \\ = O(\varepsilon^{(L+1)(1-8d/3)} + \mu^{(K+1)(1-5d/3)}), \quad (59)$$

$$|\mathcal{D}(x_i^{(Q)}) - \sin t| = O(\varepsilon^{d(2Q+3)}).$$

On the interval  $(-b\varepsilon^d, -a\varepsilon^d)$  the functions  $\chi_i, \chi_o$  have derivatives of  $O(\varepsilon^{-d})$  and second derivatives of  $O(\varepsilon^{-2d})$ . These are multiplied by the difference of the (truncated) asymptotic expansions of  $x_o(t)$ ,  $x_i(t)$  in this interval of  $t$ . As we have seen, the coefficients  $a_{klq}$  of these expansions are identical. The difference may then be estimated to be:

$$\Delta(Q, K, L, t) \equiv |(x_o - x_i)(t)| \\ \leq \sum_{q > Q, k \leq K, l \leq L} a_{klq} t^{2q+1/3} \varepsilon^k \mu^l \\ + \sum_{q \leq Q} \varepsilon^{3q/4+1/8} u_q(\tau)$$

<sup>6</sup>If  $\gamma = 0$ , then  $K$  must be set equal to 0.



where  $u_q(\tau)$  are the “rest” functions introduced in (53). Letting  $\tau = t/\varepsilon^{3/8}$  this expression is evaluated at  $t = \varepsilon^d$  to be:

$$\begin{aligned} \Delta(Q, K, L, \varepsilon^d) \\ \leq C(K, L) \varepsilon^{(2Q+1/3)d} \\ + C(Q) \varepsilon^{L(1-8d/3)} (\mu^K \varepsilon^{-5Kd/3}) \varepsilon^{1/2-d} \end{aligned}$$

where  $C(K, L), C(Q)$  are constants which depend on the truncation points, but not on  $\varepsilon$ . Similar estimates are valid for the first and second derivatives of the difference  $\Delta(Q, K, L, t = \varepsilon^d)$ . Clearly, it is possible to choose  $Q, K, L$  such that even  $\Delta \varepsilon^{-2d}$  vanishes as  $\varepsilon \rightarrow 0$  (so as to take into account the derivatives of the functions  $\chi_i, \chi_o$  in (58)) so that we may state, using the notation of (46):

**Lemma 3.3.** *The approximant  $x_a$  of (58) satisfies:*

$$\sup_{-\pi/2 < t < 0} |\mathcal{D}(x_a) - \sin t| < c_0(K, L, Q) \varepsilon^{c_1 P} \quad (60)$$

where  $P = \min(K, L, Q)$  and  $c_0, c_1$  are positive constants.

With this we can now show the existence of a solution  $X_L(t)$  of (41) which “interpolates” between the outer expansion (43) and the inner expansion (49): it is approximated by  $x_a(t)$  uniformly on  $-\pi/2 < t < 0$ . We write:

$$X_L(t) = x_a(t) + r(t)$$

and require  $X_L(-\pi/2) = x_a(-\pi/2)$ ,  $\dot{X}_L(-\pi/2) = \dot{x}_a(-\pi/2)$ . Then  $r(t)$  is the solution of the integral equation:

$$\begin{aligned} r(t) = \frac{1}{\varepsilon} \int_{-\pi/2}^t \exp\left(-\frac{\mu}{\varepsilon}(t' - t)\right) \\ \times (v_1(t')v_2(t) - v_1(t)v_2(t'))(\mathcal{D} - \sin t)(x_a)(t) \\ + 3x_a r^2 + r^3 dt' \end{aligned} \quad (61)$$

where  $v_{1,2}(t) \exp(-\mu/\varepsilon)(t + \pi/2)$  are solutions of the variational equation around  $x_a(t)$  and  $\mathcal{D}$  is the Duffing operator (46). We need here only rough bounds on these solutions. For  $t < -\varepsilon^{3/8-d}$  (for some  $0 < d < 3/8$ ) these solutions are well approximated by “WKB formulae” and may be chosen such that they have a limit as  $\varepsilon \rightarrow 0$  (see Sec. 5.3 and Appendix C) for  $t = O(\varepsilon^{3/8})$ . It turns out that, if the solutions are chosen to be  $O(1)$  at  $t = 0$ , then they are  $O(\varepsilon^{1/16})$  at  $t = -\pi/2$ . The derivative with

respect to  $\tau$  is then  $O(1)$  at  $t = 0$  but  $O(\varepsilon^{-1/16})$  at  $t = -\pi/2$ . With this, using the bounds (60) one shows in a well-known manner that, if the integer  $P$  is sufficiently large, Eq. (61) admits a unique solution of magnitude  $\sup_{-\pi/2 < t < 0} |r(t)| < C \varepsilon^{c_1 P - 1}$ , which can be obtained by iteration. We can thus conclude this section by:

**Theorem 3.1.** *Equation (41) admits a solution  $X_L(t; \varepsilon; K, L, Q)$  uniformly approximated to  $O(\varepsilon^{c_1 P - 1})$  on  $-\pi/2 < t < 0$  by  $x_a(t)$ , Eq. (58) and which obeys:  $X_L(-\pi/2) = x_a(-\pi/2)$ ,  $dX_L/dt \times (-\pi/2) = dx_a/dt(-\pi/2)$ .*

The estimates above are very rough and distort the numerical simplicity of the solution  $X_L$ : the dependence on  $K, L, Q$  is numerically irrelevant: for all practical purposes

$$\begin{aligned} X_L(t) &= (\sin t)^{1/3}, \quad -\frac{\pi}{2} < t < -\varepsilon^d; \\ &= \varepsilon^{1/8} \eta_{00}(\tau), \quad -\varepsilon^d < t < 0 \end{aligned} \quad (62)$$

(with much freedom in the choice of  $0 < d < 3/8$ ). The fact that the proof relies on the possibility to choose the integer  $P$  large originates from its ignoring the “destructive” action of the rapidly oscillating functions  $v_{1,2}(t)$ , which is apparent in their “WKB” form. It is this very weak dependence on the cutoff parameters  $(K, L, Q)$  (and thus on  $\varepsilon, \mu$ ) which justifies calling  $X_L(t)$  the (left hand) reference solution. Its behavior near  $t = 0$  is shown in Fig. 2. The figure is virtually independent of  $\varepsilon, \mu$  if time and magnitude are scaled appropriately.

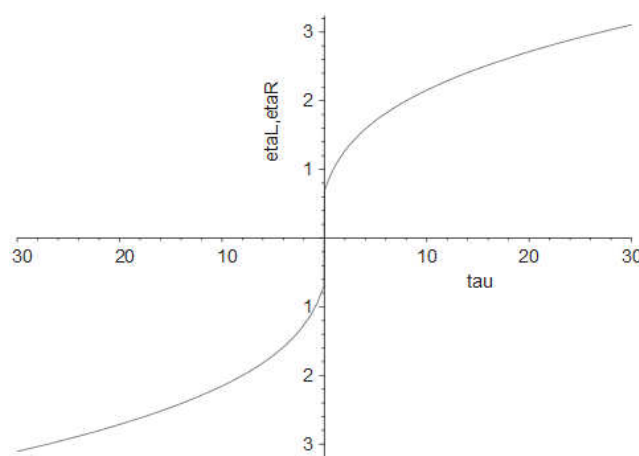


Fig. 2. The solutions  $\eta_{L,R}(\tau) = \varepsilon^{-1/8} X_{L,R}(t)$  near  $\tau = 0$ .

## 3.2. Right-hand reference solution

### 3.2.1. Choice of the inner expansion

We turn now to the interval  $0 < t < \pi/2$ . Obviously the outer expansion (43) is formally the same, with the same coefficients  $a_{klq}$ . Changing to the “inner” variables  $\tau, \eta$  we consider again an expansion in terms of  $\varepsilon^{3/4}$  similar to (49) and expect the asymptotic form of its various terms to reproduce the coefficients  $a_{klq}$ , appropriately rearranged. As will be apparent, the result is not the continuation to  $\tau > 0$  of (49). There appears now an ambiguity in the definition of (49): the boundary condition that the solutions  $\eta_q(\tau)$  of (50a), (50b), etc. should behave like  $\tau^{2q+1/3}$  as  $\tau \rightarrow \infty$  does not select a unique solution but is now obeyed by *all solutions* as a consequence of the damping term. The damping time is  $1/\gamma \approx 1/(\varepsilon^{3/8} \ln 1/\varepsilon)$ , which is shorter than the  $\tau$ -duration of  $O(1/\varepsilon^{3/8})$  of a quarter period. Equations (50a), (50b), etc. however admit solutions with almost no oscillations even at times  $\tau < 1/\gamma$ : we write to this end, in (50a) (i.e.  $q = 0$ ), for some integer  $r$  and appending an index  $R$  (for *right-hand side*):

$$\begin{aligned}\eta_{0R}(\tau) &= \sum_{k=0}^r \eta_{0kR}(\tau) \gamma^k + \gamma^{r+1} v(\tau) \\ &\equiv \eta_{0R}^{(r)} + u_0(\tau)\end{aligned}\quad (63)$$

where  $\eta_{0kR}(\tau)$  ( $k = 0, 1, \dots$ ) are, in turn, the solutions behaving like  $\tau^{1/3-5k/3}$  as  $\tau \rightarrow \infty$  of:

$$\frac{d^2 \eta_{00R}}{d\tau^2} + \eta_{00R}^3 = \tau \quad (64a)$$

$$\frac{d^2 \eta_{01R}}{d\tau^2} + 3\eta_{00R}^2 \eta_{01R} = -2 \frac{d\eta_{00R}}{d\tau}, \dots \quad (64b)$$

$$\begin{aligned}\frac{d^2 \eta_{02R}}{d\tau^2} + 3\eta_{00R}^2 \eta_{02R} \\ = -2 \frac{d\eta_{01R}}{d\tau} - 3\eta_{00R} \eta_{01R}^2, \dots\end{aligned}\quad (64c)$$

These equations are obtained by equating the coefficients of various powers of  $\gamma$  after substituting (63) in (50a). Now the condition on the behavior for  $\tau \rightarrow \infty$  selects a unique solution because the damping term is absent.<sup>7</sup> It is easy to verify that the algorithm to obtain iteratively the asymptotic

expansion of the solutions  $\eta_{0kR}$  leads indeed to:

$$\eta_{0kR} \approx \sum_l a_{kl0} \tau^{1/3-5k/3-8l/3} \quad (65)$$

with  $a_{kl0}$  of (45). The function  $u_0(\tau)$  is the solution of  $O(\gamma^{r+1})$  of the equation:

$$\begin{aligned}\frac{d^2 u_0}{d\tau^2} + 2\gamma \frac{du_0}{d\tau} + 3(\eta_0^{(r)})^2 u_0 + 3\eta_0^{(r)} u_0^2 + u_0^3 \\ = O(\gamma^{r+1} \tau^{1/3-5(r+1)/3}).\end{aligned}\quad (66)$$

Such a solution may be obtained by iteration, repeating the argument of (54).

The equations for

$$\eta_{qR}(\tau) \equiv \eta_{q0R}(\tau) + \gamma \eta_{q1R}(\tau) + \gamma^2 \eta_{q2R}(\tau) + \dots \quad (67)$$

are linear and similar to (64b), (64c), deduced from (50b), (50c) expanding in powers of  $\gamma$  and setting appropriate boundary conditions at  $\tau \rightarrow \infty$ .

### 3.2.2. The complete right-hand reference solution

We can now repeat the argument of Sec. 3.1 and obtain a solution for  $0 < t < \pi/2$  from a superposition like (58):

$$\begin{aligned}x_{aR}(t) &\equiv \chi_{oR}(t, \varepsilon^d) x_{oR}^{(Q,K,L)}(t) \\ &\quad + \chi_{iR}(t, \varepsilon^d) x_{iR}^{(Q,K)}(t)\end{aligned}\quad (68)$$

where we have now appended an index  $R$  to the various terms,

$$x_{iR}^{(Q,K)} = \varepsilon^{1/8} \left( \eta_{0R}^{(K)}(\tau) + \varepsilon^{3/4} \sum_{1 \leq q \leq Q} \eta_{qR}^{(K)}(\tau) \right) \quad (69)$$

and  $\eta_{qR}^{(K)}$  are the sums (67) restricted to  $K$  terms. Repeating the steps following Eq. (59) we state directly:

**Lemma 3.4.** *The approximant  $x_{aR}(t)$  of (68) satisfies:*

$$\sup_{0 < t < \pi/2} |\mathcal{D}(x_{aR}) - \sin t| = O(\varepsilon^{c_R P}) \quad (70)$$

with  $P = \min(K, L, Q)$  and  $c_R$  is an ( $\varepsilon$ -independent) constant.

<sup>7</sup>We do not give an explicit proof of this, because the paper contains many similar arguments.



We would like now to repeat the procedure leading to the integral equation (61): there is, however, a difficulty because a direct analogy to (61) would require an integration *backwards* in time starting at  $t = \pi/2$ . The exponential term increases in this case indefinitely as  $\varepsilon \rightarrow 0$  and precludes our setting bounds on  $r(t)$ . We have to start the integration at  $t = 0$  and set:

$$\begin{aligned} X_R(t) &= x_{aR}(t) + r(t), \quad X_R(0) = x_{iR}(0), \\ \frac{dX_R}{d\tau}(0) &= \frac{dx_{iR}}{dt}(0). \end{aligned} \quad (71)$$

The values  $x_{iR}(0)$  are obtained from the solutions of the equations for  $\eta_{qkR}$ ,  $q \leq Q$ ,  $k \leq K$  (cf. (64a), (64b), (64c), ...):

$$X_R(0) = \varepsilon^{1/8} \sum_{0 \leq q \leq Q, k \leq K} \varepsilon^{3q/4} \gamma^k \eta_{qkR}(0) \quad (72)$$

and similarly for  $dX_R/d\tau(0)$ . We conclude this section by stating in analogy to Theorem 3.1:

**Theorem 3.2.** Equation (41) admits a solution  $X_R(t; \varepsilon; K, L, Q)$  uniformly approximated to  $O(\varepsilon^{\varepsilon^R P})$  on  $0 < t < \pi/2$  by  $x_{aR}(t)$ , Eq. (68) and which obeys:  $X_R(0) = x_{aR}(0)$ , (Eq. (72))  $dX_R/dt(0) = dx_{aR}/dt(0)$ .

### 3.2.3. The discontinuity at $t = 0$

We evaluate now  $X_L(0)$ ,  $dX_L(0)/d\tau$  using (50). The correction  $r(t)$  obtained from (61) may be made as small as one wishes, by letting the cutoff integers  $K, L, Q$  be large enough. Then it is true that:

$$X_L(0) = \varepsilon^{1/8} \sum_{q \leq Q} \varepsilon^{3q/4} \eta_q(0) \equiv \varepsilon^{1/8} \eta_L(0) \quad (73)$$

where  $\eta_q(\tau)$  are the uniquely defined solutions of (50a), (50b), (50c). These solutions may be expanded in powers of  $\gamma$ , similarly to (63) (we append from now on an index  $L$  for symmetry):

$$\eta_q(\tau) \equiv \eta_{qL}(\tau) = \sum_{k \leq K} \gamma^k \eta_{qkL}(\tau) \quad (74)$$

where  $\eta_{qkL}$  verify the same Eqs. (64a)–(64c) with a boundary condition (i.e. a prescribed asymptotic behavior) at  $\tau \rightarrow -\infty$  instead of  $\tau \rightarrow \infty$ . It is easy<sup>8</sup> to relate the solutions corresponding to these two

boundary conditions, which interchange  $\tau$  and  $-\tau$ . One verifies:

$$\begin{aligned} \eta_{00L}(\tau) &= -\eta_{00R}(-\tau), & \eta_{01L}(\tau) &= \eta_{01R}(-\tau) \\ \eta_{02L}(\tau) &= -\eta_{02R}(-\tau), \dots & \eta_{10L}(\tau) &= -\eta_{10R}(-\tau) \end{aligned} \quad (75)$$

so that:

$$\begin{aligned} \Delta x_i(0) &= \varepsilon^{1/8} \sum_{q,k} \varepsilon^{3q/4} \gamma^k \eta_{qkL}(0) (1 - (-1)^{k+1}) \\ &= 2\varepsilon^{1/8} \sum_{q,k=2p} \varepsilon^{3q/4} \gamma^k \eta_{qkL}(0) \end{aligned} \quad (76)$$

and

$$\Delta \frac{dx_i}{dt}(0) = 2\varepsilon^{1/8} \sum_{q,k=2p+1} \varepsilon^{3q/4} \gamma^k \frac{d\eta_{qkL}}{d\tau}(0). \quad (77)$$

Since  $\eta_{00L}(0) \neq 0$ , it follows that  $X_L(0) \neq X_R(0)$ , thus the two *reference* solutions are *not* the continuation of each other. There is a jump of  $O(\varepsilon^{1/8})$  at  $t = 0$  in going from one to the other. The derivatives have a smaller jump of  $O(\varepsilon^{1/8}\gamma)$ . We call these solutions “*reference*” solutions because the motions which we study consist of small oscillations around them. Eilenberger and Schmidt call them *creeping solutions*: the left-hand solution  $X_L(t)$  is the motion of a particle which stays at the bottom of the moving potential well  $V(x) = x^4/4 - x \sin t$  for all times away from  $t = 0$ ; near  $t = 0$ , the velocity of the minimum of the well at  $x_m(t) = (\sin t)^{1/3}$  becomes unbounded and the particle cannot follow it: for  $t > 0$  it will oscillate around the minimum with a larger amplitude (see Sec. 6.1). It behaves as if it had received a *kick* at  $t = 0$ .<sup>9</sup> In order that the particle follow the minimum of the potential for  $t > 0$  away from zero, it has to start at  $t = 0$  from  $(X_R(0), dX_R/dt(0))$ : this is the *right-hand creeping solution*: it will stay near the minimum until  $t \approx 3\pi/2$ . In Fig. 2 we show the appearance of the reference solutions  $\eta_R, \eta_L$  near  $t = 0$  ( $\eta_{R,L} = \varepsilon^{-1/8} X_{R,L}$ ).

### 3.3. The Poincaré map

We define the time- $2\pi$  Poincaré map with respect to the reference solutions  $X_L(t)$ ,  $X_R(t)$  constructed

<sup>8</sup>Invoking the uniqueness of the solutions.

<sup>9</sup>However, there is no real “kick” and in my view the model proposed by the authors with a discontinuous force at  $t = 0$  is not a correct description of the appearance of the circle map for the Duffing equation.



in the previous subsections. If we choose the same cutoff parameters  $K, L$  in both, it is true that (to order  $\varepsilon^c P$ ):

$$X_L\left(-\frac{\pi}{2}\right) = -X_R\left(\frac{\pi}{2}\right),$$

$$\frac{dX_L}{dt}\left(-\frac{\pi}{2}\right) = -\frac{dX_R}{dt}\left(\frac{\pi}{2}\right).$$

As already noticed (cf. Eq. (2)), if  $x(t)$  is a solution of Eq. (41), then  $-x(t - \pi)$  is also a solution. Thus the solution starting at  $t = \pi/2$  with the values  $-X_L(-\pi/2)$ ,  $-dX_L/dt(-\pi/2)$ , is simply  $-X_L(t - \pi)$ , which we denote by  $X_{L1}(t)$ . This solution is a "reference" solution up to  $t = \pi$  where it is replaced by  $X_{R1}(t) \equiv -X_R(t - \pi)$ . We write, for a solution  $x(t)$  of (41):

$$\begin{aligned} x(t) &\equiv X_L(t) + v_L(t), & -\frac{\pi}{2} \leq t < 0, \\ x(t) &\equiv X_R(t) + v_R(t), & 0 \leq t < \frac{\pi}{2} \\ x(t) &\equiv X_{L1}(t) + v_{L1}(t), & \frac{\pi}{2} \leq t < \pi, \\ x(t) &\equiv X_{R1}(t) + v_{R1}(t), & \pi \leq t < \frac{3\pi}{2} \\ x(t) &\equiv X_{L2}(t) + v_{L2}(t), & \frac{3\pi}{2} \leq t < 2\pi, \text{ etc.} \end{aligned} \quad (78)$$

We define the *time-2 $\pi$  Poincaré map*  $\mathbb{P}_0$  by:

$$\begin{aligned} \mathbb{P}_0 : \left( v_L\left(-\frac{\pi}{2}\right), \frac{dv_L}{dt}\left(-\frac{\pi}{2}\right) \right) \\ \Rightarrow \left( v_{L2}\left(\frac{3\pi}{2}\right), \frac{dv_{L2}}{dt}\left(\frac{3\pi}{2}\right) \right). \end{aligned} \quad (79)$$

If  $x(t)$  is the unique periodic solution of (41) then it must be odd periodic and leads thus also to a fixed point of the *half-period Poincaré map*:

$$\begin{aligned} \mathbb{P} : \left( v_L\left(-\frac{\pi}{2}\right), \frac{dv_L}{dt}\left(-\frac{\pi}{2}\right) \right) \\ \Rightarrow \left( -v_{L1}\left(\frac{\pi}{2}\right), -\frac{dv_{L1}}{dt}\left(\frac{\pi}{2}\right) \right). \end{aligned} \quad (80)$$

The symmetry  $t \rightarrow t + \pi$ ,  $x \rightarrow -x$  implies that, in fact:

$$\mathbb{P}_0 = \mathbb{P} \circ \mathbb{P}. \quad (81)$$

Indeed, under  $\mathbb{P}$  the point  $P_i(x(-\pi/2), dx/dt(-\pi/2))$  corresponding to a solution  $x(t)$ , moves to

$P_f(-x(\pi/2), -dx/dt(\pi/2))$ ; Also, the point  $P_i^1(x_1(-\pi/2), dx_1/dt(-\pi/2))$  corresponding to the solution  $x_1(t) \equiv -x(t + \pi)$  moves to  $P_f^1(-x_1(\pi/2), -dx_1/dt(\pi/2))$ . But  $x_1(-\pi/2) = -x(\pi/2)$ ,  $dx_1/dt(-\pi/2) = -dx/dt(\pi/2)$ , i.e.  $P_i^1 \equiv P_f$  and  $-x_1(\pi/2) = x(3\pi/2)$ ,  $-dx_1/dt(\pi/2) = dx/dt(3\pi/2)$ . Therefore  $P_f^1$  is at the same time the image of the original point  $P_i$  under  $\mathbb{P} \circ \mathbb{P}$  and its image under  $\mathbb{P}_0$ . This shows the validity of (81). Now, the mismatch at  $t = \pi/2$  between  $-X_L(t - \pi/2)$  and  $X_R(t)$  may be made arbitrarily small, simply by increasing the truncation orders, according to Theorems 3.1 and 3.2. We shall ignore this small correction in the following and regard  $\mathbb{P}$  as being simply:

$$\begin{aligned} \mathbb{P}(\varepsilon, \mu) : \left( v_L\left(-\frac{\pi}{2}\right), \frac{dv_L}{dt}\left(-\frac{\pi}{2}\right) \right) \\ \Rightarrow \left( v_R\left(\frac{\pi}{2}\right), \frac{dv_R}{dt}\left(\frac{\pi}{2}\right) \right). \end{aligned} \quad (82)$$

In the next section, the definition will be further modified, by introducing another independent (time) variable.

## 4. The Invariant Sets of $\mathbb{P}(\cdot; \cdot)$

### 4.1. A characterization of possible invariant sets

The function  $v_L(t)$ , defined as the departure of a solution  $x(t)$  from the reference function  $X_L(t)$  (cf. Eq. (78)) obeys:

$$\varepsilon \frac{d^2 v_L}{dt^2} + 2\mu \frac{dv_L}{dt} + 3X_L^2 v_L + 3X_L v_L^2 + v_L^3 = 0. \quad (83)$$

A similar equation holds for the function  $v_R(t)$ , Eq. (78), the departure of  $x(t)$  from the right-hand side ( $t > 0$ ) reference solution  $X_R(t)$ . It is convenient to introduce new independent variables for  $t < 0$ ,  $t > 0$ , in turn:

$$\begin{aligned} \theta_L &= 3^{1/2} \varepsilon^{-1/2} \int_t^{-\tau_0 \varepsilon^{3=8}} X_L(t') dt', \\ \theta_R &= 3^{1/2} \varepsilon^{-1/2} \int_{\tau_0 \varepsilon^{3=8}}^t X_R(t') dt' \end{aligned} \quad (84)$$

for a  $\tau_0$  such that  $X_L(\tau)$ ,  $X_R(\tau) \neq 0$  for  $\tau < -\tau_0$ ,  $\tau > \tau_0$ , in turn. In this section, we prove the

following:

**Theorem 4.1.** *If the Poincaré map  $\mathbb{P}(\varepsilon, \mu)$  has invariant sets, then, for  $\varepsilon$  sufficiently small, they are contained in a rectangle:*

$$\mathbb{D}_0 : \left| v_L \left( -\frac{\pi}{2} \right) \right|, \left| \frac{dv_L}{d\theta_L} \left( -\frac{\pi}{2} \right) \right| < M \varepsilon^{3/16 + \kappa\pi/2} \quad (85)$$

with (cf. Eq. (21))

$$\kappa = \kappa(\varepsilon) = \frac{\mu}{\varepsilon \ln \left( \frac{1}{\varepsilon} \right)} = \frac{3}{2} \frac{\Delta}{\ln \Gamma} \quad (86)$$

and  $M$  independent of  $\varepsilon$ .

With this definition of  $\kappa(\varepsilon)$ , the “damping factor” reads:

$$\exp(-\Delta t) = \exp \left( -\frac{\mu}{\varepsilon} t \right) = \varepsilon^{\kappa t}. \quad (87)$$

Since (as will turn out) bifurcations occur when  $\kappa = O(1)$ , we shall use this latter notation from now on.

## 4.2. A qualitative argument

To explain intuitively the origin of this theorem, we notice first that, as a consequence of Lemma 2.2, all solutions of (83) must obey eventually

$$\begin{aligned} \left| v_L \left( -\frac{\pi}{2} \right) \right| &< K \varepsilon^{C\kappa\pi/4} \\ \left| \frac{dv_L}{dt} \left( -\frac{\pi}{2} \right) \right| &< K \varepsilon^{-1/2} \varepsilon^{C\kappa\pi/4} \end{aligned} \quad (88)$$

with  $K, C$  of (34). Indeed, the reference solution  $X_L(t)$  obeys Condition 2.1 of Sec. 2.2 on an interval  $[-3\pi/4, -\pi/4]$ : there  $X_L(t) \approx (\sin t)^{1/3}$  (cf. Eq. (62)) and one can choose  $a, b \approx 2^{-1/6}$ . Thus, invariant sets can be only subsets of the rectangle (88). Using the variable  $\theta_L$ , Eq. (84), the last inequality is transformed into (with a redefinition of  $K$ ):

$$\left| \frac{dv_L}{d\theta_L} \left( -\frac{\pi}{2} \right) \right| < K \varepsilon^{C\kappa\pi/4}.$$

With this in mind, we perform a change to a new dependent variable  $w_L(\theta_L)$  through:

$$v_L(t) = \frac{w_L(\theta_L)}{(-X_L)^{1/2}} \varepsilon^{\kappa(t+\pi/2)} \varepsilon^\alpha \quad (89)$$

where  $\alpha > 0$  is for the time being unspecified. Clearly,  $v_L(-\pi/2) = w_L(\theta(-\pi/2)) \varepsilon^\alpha$ . The function

$w_L$  verifies for  $t < 0$ :

$$\begin{aligned} \frac{d^2 w_L}{d\theta_L^2} + w_L(1 + g_L(\theta_L)) - w_L^2 h(\theta_L) \\ + \frac{h^2(\theta_L)}{3} w_L^3 = 0 \end{aligned} \quad (90)$$

where:

$$h(\theta_L) = \frac{\varepsilon^{\kappa(t+\pi/2)} \varepsilon^\alpha}{(-X_L)^{3/2}} \quad (91)$$

$$\begin{aligned} g_L(\theta_L) = \frac{\varepsilon}{4} \frac{\left( \frac{dX_L}{dt} \right)^2}{X_L^4} - \frac{\varepsilon}{6} \frac{d^2 X_L}{X_L^3} \\ - \kappa^2 \varepsilon \frac{\ln^2 \left( \frac{1}{\varepsilon} \right)}{3X_L^2}. \end{aligned} \quad (92)$$

Since  $(-X_L)(t) \approx (\sin t)^{1/3}$  for  $t < -\tau_0 \varepsilon^{3/8}$ , one checks that  $g_L(\theta_L) \approx \varepsilon(t^{-8/3} + t^{-2/3} \ln^2(1/\varepsilon))$  and is thus  $O(1)$  at  $t = O(\varepsilon^{3/8})$ , drops off at larger  $|t|$  and becomes  $O(\varepsilon \ln^2(1/\varepsilon))$  at  $t = -\pi/2$ . The function  $h(\theta_L)$  is not monotonical but has a minimum at  $t$  of  $O(1/(\kappa \ln \varepsilon))$  where it is  $O(\varepsilon^{\kappa\pi/2+\alpha} (\kappa \ln(1/\varepsilon))^{1/2})$ . It is  $O(1)$  at  $t = O(\varepsilon^{\kappa\pi+2\alpha})$ , if  $\kappa\pi + 2\alpha < 3/8$  but stays otherwise  $o(1)$  down to  $t = O(\varepsilon^{3/8})$ . We denote:

$$p \equiv \min \left( \frac{\kappa\pi + 2\alpha}{3}, \frac{1}{8} \right) \quad (93)$$

so that  $h(\theta_L)$  is  $O(1)$  at  $t \approx \varepsilon^{3p}$ , if  $p < 1/8$ . The following is a qualitative argument for:

**Statement 4.1.** *If  $p < 1/8$ , the order of magnitude of  $v_R(\pi/2)$  is  $\varepsilon^{\alpha+\kappa\pi}$  (i.e. smaller by a factor  $\varepsilon^{\kappa\pi}$  than  $v_L(-\pi/2)$ ). If  $p = 1/8$ , then  $v_R(\pi/2)$  is  $O(\varepsilon^{3/16+\kappa\pi/2})$  and this order remains unchanged in the following half periods.*

The essential point in the argument is that the magnitude of the jump of the “reference” solutions  $X_L(t), X_R(t)$  is  $O(\varepsilon^{1/8})$ . Equation (90) describes an oscillatory motion in a time( $\theta$ )-dependent potential which has a single minimum at  $w_L = 0$ . If  $|w_L| < \text{const}$  at  $t = -\pi/2$ , one expects that this motion remains bounded, uniformly with respect to  $\varepsilon$ , at least as long as  $|h| < \text{const}$ , i.e. down to times of  $O(\varepsilon^{3p})$ . At such values of  $t$ , according to (89),  $v_L(t)$  is then of  $O(\varepsilon^p)$ . The order of magnitude of  $v_L(t)$  is likely to stay unchanged down to  $t = 0$ ; there, the reference solution is changed from  $X_L(t)$  to  $X_R(t)$ , which means a shift of  $O(\varepsilon^{1/8})$ . If  $p < 1/8$ ,



this shift is unnoticed: the function  $v_R(t)$  has the same order of magnitude as  $v_L(t)$  and we shall show that this is preserved up to  $t$  of  $O(\varepsilon^{3p})$ . Using a change of variables similar to (89):

$$v_R(t) = \frac{w_R(\theta_R)}{X_R^{1/2}} \varepsilon^{\kappa t + 3p/2} \quad (94)$$

we verify that  $w_R(\theta_R)$  obeys:

$$\begin{aligned} \frac{d^2 w_R}{d\theta_R^2} + w_R(1 + g_R(\theta_R)) + w_R^2 k(\theta_R) \\ + \frac{w_R^3}{3} k^2(\theta_R) = 0 \end{aligned} \quad (95)$$

with  $g_R(\theta_R)$  analogous to  $g_L(\theta)$  of (88) and

$$k(\theta_R) = \frac{\varepsilon^{\kappa t + 3p/2}}{X_R^{3/2}(t)}. \quad (96)$$

The function  $k(\theta_R)$  is monotonically decreasing from  $O(\varepsilon^{3p/2-3/16})$  at  $t = O(\varepsilon^{3/8})$  to  $O(\varepsilon^{\kappa\pi/2+3p/2})$  at  $t = \pi/2$ . Now, if at  $t = O(\varepsilon^{3p})$ ,  $v_R(t) = O(\varepsilon^p)$ , it follows that  $w_R(\theta_R)$  is  $O(1)$  there and, since the motion described by (95) is oscillatory and — hopefully — bounded, it stays so up to  $t = \pi/2$ ; consequently,  $v_R(\pi/2) = O(\varepsilon^{\kappa\pi/2+3p/2})$ . This is  $O(\varepsilon^{\kappa\pi+\alpha})$  if  $p < 1/8$ . Thus, in this situation we start the next half period with a value of  $v_L(-\pi/2 + \pi)$  damped with respect to the original one by a factor  $\varepsilon^{\kappa\pi}$ . As announced in Statement 4.1, if  $p < 1/8$ , the new half cycle starts with a value of  $\alpha$ , increased by  $\kappa\pi$ . After a finite number of cycles,  $\alpha$  will be such that the inequality  $p < 1/8$  is no longer valid. When this occurs, the magnitude of  $v_L(t)$  at  $t = 0$  is  $O(\varepsilon^{1/8})$  and, since the jump of the reference solutions is  $O(\varepsilon^{1/8})$ ,  $v_R(t)$  is also of  $O(\varepsilon^{1/8})$ . Then, the change of variables (94) with  $p = 1/8$  shows that  $w_R(\theta)$  is  $O(1)$  when  $t$  is of  $O(\varepsilon^{3/8})$  and it follows that  $v_R(\pi/2)$  is of  $O(\varepsilon^{3p/2}) = O(\varepsilon^{\kappa\pi/2+3/16})$ , as stated in Theorem 4.1.

To conclude, if we start with a value  $\alpha$  such that  $p < 1/8$ , i.e.  $\alpha < 3/16 - \kappa\pi/2$ , it will increase in the following half cycles until it gets over this bound; in the succeeding half cycles it does not get any more below. Indeed, if we start with  $\alpha \geq 3/16 - \kappa\pi/2 \equiv \alpha_0$ ,  $h(t)$  is  $O(\varepsilon^s)$ , with  $s \geq 0$  at  $(-t)$  of  $O(\varepsilon^{3/8})$  and  $v(t)$  is of  $O(\varepsilon^{1/8+s})$ ; it is thus at least a factor  $\varepsilon^s$  smaller than the jump of the reference solutions. The motion continues at  $t > 0$  with oscillations of  $O(\varepsilon^{1/8})$  around the reference  $X_R(t)$ ; their amplitude decreases gradually (due to the factor  $X_R^{-1/2}$ )

and, as a consequence of the damping, becomes  $O(\varepsilon^{\kappa\pi/2+3/16})$  at  $t = \pi/2$ . Thus, once solutions are “trapped” in a rectangle (85), they stay there all the time.

For a proof of Theorem 4.1 (and of Statement 4.1), one has to place indeed bounds independent of  $\varepsilon$  on the magnitude of the solutions  $w_{L,R}(\theta)$  of Eqs. (90), (93) and also justify the preservation of the magnitude of the solutions in the transition region  $(-\tau_0\varepsilon^p, \tau_0\varepsilon^p)$ .

#### 4.3. The interval $-\pi/2 < t < \pi/2$

We show that, if the energy of the oscillations of  $w_L$  is bounded at  $t = -\pi/2$ , it stays bounded up to  $t = -\tau_0\varepsilon^{3p}$ , uniformly with respect to  $\varepsilon$ , if  $\tau_0$  is sufficiently large. To this end, we consider the energy associated with (90):

$$\begin{aligned} E(\theta) = \frac{1}{2} \left( \frac{dw_L}{d\theta_L} \right)^2 + \frac{1}{2} w_L^2 \left[ (1 + g(\theta_L)) \right. \\ \left. - \frac{2}{3} w_L h(\theta_L) + \frac{1}{6} w_L^2 h(\theta_L)^2 \right]. \end{aligned} \quad (97)$$

The quantity in square brackets is positive definite and larger than  $1/3 + g(\theta_L)$ , so that  $(g(\theta_L) > 0)$ :

$$\begin{aligned} |w_L(\theta_L)| &< \sqrt{6E(\theta_L)}, \\ \left| \frac{dw_L}{d\theta_L}(\theta_L) \right| &< \sqrt{2E(\theta_L)}. \end{aligned} \quad (98)$$

It follows that, if  $E > 1$  and for those values of  $\theta_L$  for which  $|h(\theta_L)| < 1$

$$\begin{aligned} \left| \frac{dE}{d\theta_L} \right| &= \left| \frac{1}{2} w_L^2 \left[ \frac{dg}{d\theta_L} - \frac{2}{3} w_L \frac{dh}{d\theta_L} + \frac{1}{3} w_L^2 h \frac{dh}{d\theta_L} \right] \right| \\ &< 3E(\theta_L)^2 \left( \left| \frac{dg}{d\theta_L} \right| + 6 \left| \frac{dh}{d\theta_L} \right| \right). \end{aligned} \quad (99)$$

This inequality implies that, if the energy at  $-\pi/2$  is bounded by a number  $E_0$  and if  $\tau_{0L}(E_0)$  is such that  $h(\tau_{0L}\varepsilon^{3p})$  and  $g(\tau_{0L}\varepsilon^{3p})$  are so small that:

$$\frac{1}{E_0} > 3g(\tau_{0L}\varepsilon^{3p}) + 6h(\tau_{0L}\varepsilon^{3p})$$

then  $E(\tau_{0L}\varepsilon^{3p})$  is bounded. These inequalities imply that  $|v_L(\tau_{0L}\varepsilon^{3p})|$ ,  $|dv_L/d\theta_L(\tau_{0L}\varepsilon^{3p})|$  are bounded by  $C(E_0)\varepsilon^p$  at these values of  $t$ . The constant increases as  $E_0$  increases.



#### 4.4. The interval $(-\tau_{0L}\varepsilon^{3p}; \tau_{0R}\varepsilon^{3p})$

Changing variables to

$$t = \sigma\varepsilon^{1/2-p}, \quad v_L = V\varepsilon^p \quad (100)$$

one transforms (83) to:

$$\begin{aligned} \frac{d^2V}{d\sigma^2} + 2\kappa\varepsilon^{1/2-p}\ln\left(\frac{1}{\varepsilon}\right)\frac{dV}{d\sigma} + 3(X_L^2\varepsilon^{-2p})V \\ + 3(X_L\varepsilon^{-p})V^2 + V^3 = 0. \end{aligned} \quad (101)$$

For  $t$  in  $(-\tau_{0L}\varepsilon^{3p}, 0)$ , it is true that  $|X_L\varepsilon^{-p}| < \tau_{0L}^{1/3}$ . Also, one verifies that, at  $t = -\tau_{0L}\varepsilon^{3p}$ ,  $|dV/d\sigma|$  is  $O(1)$ . Thus the energy associated to (101) is bounded by a constant at  $t = -\tau_{0L}\varepsilon^{3p}$ . Further, it is true that, since the potential function is bounded from below by  $V^4/6$ ,

$$|V(\sigma)| < 2E^{1/4}. \quad (102)$$

Then, assuming  $E > 1$ , we may bound

$$\frac{dE}{d\sigma} < 8E^{3/4} \left( \frac{3}{2} \left| \frac{d(X_L\varepsilon^{-p})^2}{d\sigma} \right| + \left| \frac{dX_L\varepsilon^{-p}}{d\sigma} \right| \right)$$

and, integrating this inequality from  $-\tau_{0L}\varepsilon^{3p}$  to  $0^{10}$ :

$$E(0)^{1/4} < E(-\tau_{0L}\varepsilon^{3p})^{1/4} + 2 \left( \frac{3}{2}\tau_{0L}^{2/3} + \tau_{0L}^{1/3} \right). \quad (103)$$

It follows that both  $|V|$  and  $|dV/d\sigma|$  are bounded at  $t = 0$  and therefore:

$$|v_L(0)| < C\varepsilon^p, \quad \left| \frac{dv_L}{d\sigma}(0) \right| < C\varepsilon^p. \quad (104)$$

With our definition (93) of  $p$ , the departure  $v_R(t)$  of  $x(t) = X_L(t) + v_L(t)$  from  $X_R(t)$  is also of  $O(\varepsilon^p)$  and the same is true for  $dv_R/d\sigma(0)$ . With the change of variables (100), with  $v_L$  replaced by  $v_R$ , we obtain an equation identical to (101) with  $X_R(t)$  instead of  $X_L(t)$ . The same argument as before shows that the energy associated to it is bounded at  $t = \tau_{0R}\varepsilon^{3p}$ , for some (at this stage arbitrary) choice of  $\tau_{0R}$ . It follows that  $|v_R(\tau_{0R}\varepsilon^{3p})|$ ,  $|dv_R/d\theta_R(\tau_{0R}\varepsilon^{3p})|$  are bounded by  $C(\tau_{0R})\varepsilon^p$ . Clearly, the value of  $C(\tau_{0R})$  increases with  $\tau_{0R}$ .

#### 4.5. The interval $(\tau_{0R}\varepsilon^{3p}; \tau_{0R}\varepsilon^q)$ ; $q < 3p$

In this interval  $\varepsilon^{\kappa t} \approx 1$  and we may write:

$$\begin{aligned} k(\theta_R) &\approx \frac{1}{\tau_{0R}^{1/2}} \left( \frac{\theta_{0R}}{\theta_R} \right)^{3/8}, \\ \theta_{0R} &\approx \frac{3\sqrt{3}}{4} \varepsilon^{4p-1/2} \tau_{0R}^{4/3}. \end{aligned} \quad (105)$$

The range of values of  $\theta_R$  corresponding to this interval may become arbitrarily large if  $\varepsilon$  is chosen sufficiently small. We define  $w_R$  through Eq. (94): it follows from the previous paragraph that  $|w_R|$ ,  $dw_R/d\theta_R$  are  $O(1)$  at  $t = \varepsilon^{3p}$ . We shall show that  $w_R$  and  $dw_R/d\theta_R$  stay bounded in the whole interval  $(\varepsilon^{3p}, \varepsilon^q)$ .  $w_R(\theta_R)$  obeys Eq. (95) and by analogy to (97), with the same notation for the energy, but the replacement of  $h(\theta_L)$  with  $-k(\theta_R)$  we may write:

$$\frac{dE}{d\theta_R} = \frac{1}{2}w_R^2 \frac{dg_R}{d\theta_R} + \frac{w_R^3}{3} \frac{dk}{d\theta_R} + \frac{1}{6}w_R^4 k \frac{dk}{d\theta_R}. \quad (106)$$

We cannot repeat the argument of Sec. 4.3 because, if we try to increase the value of  $\theta_{0R}$  to ensure an inequality like (99) we increase at the same time the bound on the possible energies (see the last paragraph of Sec. 4.4). One seems to need a rather long detour.

In (105),  $dg_R/d\theta_R$ ,  $dk/d\theta_R$  are strictly negative, so that the only term which may change sign is the middle one with  $w_R^3$ . Thus:

$$\frac{dE}{d\theta_R} < \frac{1}{3}|w_R^3| \left| \frac{dk}{d\theta_R} \right|. \quad (107)$$

We can use now a bound on  $w_R$  similar to (98) to derive the inequality<sup>11</sup>:

$$\frac{dE}{d\theta_R} < CE^{3/2}(\theta_R) \left| \frac{dk}{d\theta_R} \right|. \quad (108)$$

Unfortunately, we cannot draw any conclusions about the boundedness of  $E$  for large  $\theta_R$  directly from (108) unless some other restriction on  $E(\theta_R)$  holds. We shall show in Appendix A that, in fact,  $E(\theta_R)$  increases for large  $\theta_R$  at most like  $\theta_R^{3/4-s}$  for some  $s > 0$ , i.e. for  $\theta_R$  in this time interval:

$$E(\theta_R) < \text{const} \left( \frac{\theta_R}{\theta_{0R}} \right)^{3/4-s}. \quad (109)$$

<sup>10</sup>If  $p = 1/8$  there are also constant contributions of the upper limit  $t = 0$ ; they do not play a significant part.

<sup>11</sup>Capital  $C$  denotes a constant which need not be specified in more detail.

Using this in (108) we obtain:

$$\frac{dE}{d\theta_R} < C \left( \frac{\theta_R}{\theta_{0R}} \right)^{9/8-3s/2} \left( \frac{\theta_{0R}}{\theta_R} \right)^{3/8} \frac{1}{\theta_R} \quad (110)$$

which leads by integration to an improved bound on  $E(\theta_R)$ :

$$E(\theta_R) - E(\theta_{0R}) < C \left( \frac{\theta_R}{\theta_{0R}} \right)^{3/4-9s/4}. \quad (111)$$

Using this bound again in (108), we can further improve (111) and after a finite number of such steps, the power of  $(\theta_R/\theta_{0R})$  decreases enough so that we can state:

$$E(\theta_R) < \text{const} \quad (112)$$

for all  $\theta_R$  in the time interval  $(\varepsilon^{3p}, \varepsilon^q)$ .

#### 4.6. The interval $(\varepsilon^q, \varepsilon^2)$

This time integration of (108) leads directly to the desired bound:

$$\frac{1}{\sqrt{E(\theta_R)}} > \frac{1}{\sqrt{E(\theta_R(\varepsilon^q))}} - 2C(k(\theta_R(\varepsilon^q)) - k(\theta_R)) \quad (113)$$

we can always choose  $\varepsilon$  so small that the right-hand side be positive.<sup>12</sup> This leads then to an upper bound on  $E(\theta_R)$  and thus on  $E(\pi/2)$ , as announced.

#### 4.7. Summary

The important point concerning the bounds which were established above is that they are independent of  $\varepsilon$ , only provided  $\varepsilon$  is sufficiently small. We can now review the qualitative argument for Theorem 4.1 given in Sec. 4.2, whose gaps have now been filled in. As we have seen *all* solutions reach at some time  $n\pi/2$  the interior of a rectangle  $\mathbb{D}$  of size  $K\varepsilon^{C\kappa\pi/4}$  around the reference solution<sup>13</sup> for a certain constant  $K$ . If there are invariant sets, they are contained in this rectangle for all times  $m\pi/2$ ,  $m = -\infty \cdots \infty$ . If  $C\kappa\pi/4 > 3/16 - \kappa\pi/2 \equiv \alpha_0$  then the argument of Sec. 4.2 shows that those solutions that are in  $\mathbb{D}$  at, say,  $t = -\pi/2$ , are contained at  $t = \pi/2$  in the rectangle  $\mathbb{D}_0$  of size  $M_1\varepsilon^{\kappa\pi/2+3/16}$ , Eq. (85) which proves Theorem 4.1 for this situation. If the inequality is not satisfied, we may find a constant  $\bar{M}_1$  so that the solutions are contained at

$t = \pi/2$  in a rectangle  $\mathbb{D}_1$  of size  $\bar{M}_1\varepsilon^{\kappa(\pi+C\pi/4)}$ , as argued in Sec. 4.2. If  $\kappa(C\pi/4 + \pi) > \alpha_0$ , then the solutions reach at  $t = 3\pi/2$  the interior of  $\mathbb{D}_0$  and the proof stops at this stage. If not, we find  $\bar{M}_2$  and a rectangle  $\mathbb{D}_2$  of size  $\bar{M}_2\varepsilon^{\kappa(C\pi/4+2\pi)}$  so that the solutions are contained within it at  $t = 3\pi/2$ . After a finite number of steps, the bound  $\alpha_0$  will be overcome and this ends the proof of Theorem 4.1.

Independently of the existence of invariant sets, the arguments of this section lead to the following:

**Corollary 4.1.** *Consider the solutions  $v_L(t)$  starting at  $t = -\pi/2$  in the rectangle (85) of Theorem 4.1 and the corresponding functions  $w_L(\theta)$ ,  $w_R(\theta)$  defined in (89) and (94). There exists then a constant  $M$ , independent of  $\varepsilon$  if  $\varepsilon$  is sufficiently small, so that*

$$|w_{L,R}(\theta_{L,R})|, \quad \left| \frac{dw_{L,R}}{d\theta_{L,R}} \right| < M \quad (114)$$

for  $t(\theta_{L,R})$  in  $(-\pi/2, -\tau_0\varepsilon^{3/8})$  and  $(\tau_0\varepsilon^{3/8}\pi/2)$ , in turn.

This remark is used in the next sections to justify the averaging procedures employed there (see [Arnold, 1978, Section 52]).

### 5. The Left-Hand Side Poincaré Map

#### 5.1. The left-hand quarter period map

In this and the next section we derive approximations to the half period Poincaré map  $\mathbb{P}_L$  restricted to the rectangle (85) of Theorem 4.1. We consider first the quarter period Poincaré map:

$$\begin{aligned} \mathbb{P}_L : \left( v_L \left( -\frac{\pi}{2} \right), \frac{dv_L}{d\theta_L} \left( -\frac{\pi}{2} \right) \right) \\ \Rightarrow \left( v_L(0), \frac{dv_L}{d\tau}(0) \right) \end{aligned} \quad (115)$$

where  $\tau$  is the “boundary layer” time variable introduced in Sec. 1, Eq. (18),  $\theta_L$  is defined in Eq. (84) and  $v_L$  is a solution of Eq. (83) for  $-\pi/2 < t < 0$ . **Note:** In this section we shall drop the index “L” on the variable  $\theta$  because  $\theta_R$ , Eq. (84), used for  $t > 0$ , does not appear at all. Also, for ease of notation, we write  $g(\theta) \equiv g_L(\theta)$  of (92). Instead of the rectangle

<sup>12</sup>The value  $E(\theta_R(\varepsilon^q))$  is independent of  $\varepsilon$ , if  $\varepsilon$  is sufficiently small, according to Sec. 4.5.

<sup>13</sup>If the variable  $\theta_{L,R}$  is used for time.



Eq. (85) we may consider a disk of the same magnitude and parametrize:

$$\begin{aligned} v_L \left( -\frac{\pi}{2} \right) &= \varepsilon^{3/16+\kappa\pi/2} \Lambda \cos \Psi_0 \\ \frac{dv_L}{d\theta} \left( -\frac{\pi}{2} \right) &= -\varepsilon^{3/16+\kappa\pi/2} \Lambda \sin \Psi_0. \end{aligned} \quad (116)$$

With the change of dependent variables (89), we are led to Eq. (90) where now  $\alpha = 3/16 + \kappa\pi/2$ . The function  $h(\theta)$  of (91) is  $O(\varepsilon^{3/16+\kappa\pi/2})$  at  $t = -\pi/2$  and  $O(\varepsilon^{\kappa\pi})$  at  $t = O(\varepsilon^{3/8})$ . In an interval  $I_L \equiv (-C_\varepsilon, -\tau_0\varepsilon^{3/8})$ , where  $C_\varepsilon \rightarrow 0$  when  $\varepsilon \rightarrow 0$ , we can set  $\sin t \approx t$ , so that Eq. (84) implies  $t \approx -\varepsilon^{3/8}(-\theta)^{3/4}$ , and it follows that, for  $|\tau|$  sufficiently large:

$$\begin{aligned} h(\theta) &\equiv \frac{\varepsilon^{\kappa(t+\pi/2)} \varepsilon^{\kappa\pi/2+3/16}}{(-X_L)^{3/2}} \\ &\approx \varepsilon^{\kappa\pi} \left( \frac{-\theta_0}{-\theta} \right)^{3/8}, \quad \theta_0 = -\frac{3^{3/2}}{4}. \end{aligned} \quad (117)$$

For<sup>14</sup>  $\varepsilon^{3/8} = 0.003$ , Fig. 3 shows the appearance of  $h(t)$ ; in the range of  $\kappa$  values of interest (see below), the function  $g(t)$  is much smaller than  $h(t)$  (at  $t = -10 \times \varepsilon^{3/8}$  it is  $\approx 8.5 \times 10^{-5}$ ).

## 5.2. The interval ( $\varepsilon = 2; \varepsilon^{3/8}$ )

The influence of the nonlinear terms of (90) can be analyzed due to the smallness of the function  $h(\theta)$  in a manner inspired by and related to the

averaging method of Bogolyubov and Mitropolsky [1961] (especially Chapter V). We use new dependent variables  $R(\theta)$  and  $\phi(\theta)$ :

$$w_L(\theta) = R(\theta) \cos(\theta - \theta_0 + \phi(\theta)) \quad (118a)$$

$$\frac{dw_L}{d\theta} = -R(\theta) \sin(\theta - \theta_0 + \phi(\theta)) \quad (118b)$$

which transform (90) into the pair of equations:

$$\begin{aligned} \frac{dR}{d\theta} &= \frac{1}{2}g(\theta)R(\theta) \sin(2z) \\ &\quad - \frac{1}{4}h(\theta)R(\theta)^2(\sin z + \sin(3z)) \\ &\quad + \frac{1}{12}h(\theta)^2R(\theta)^3 \left( \sin(2z) + \frac{1}{2} \sin(4z) \right) \end{aligned} \quad (119a)$$

$$\begin{aligned} \frac{d\phi}{d\theta} &= \frac{1}{2}g(\theta)(1 + \cos(2z)) \\ &\quad - \frac{1}{4}h(\theta)R(\theta)(3 \cos z + \cos(3z)) \\ &\quad + h(\theta)^2R(\theta)^2 \left( \frac{3}{2} + 2 \cos(2z) + \frac{1}{2} \cos(4z) \right) \end{aligned} \quad (119b)$$

with

$$z(\theta) = \theta - \theta_0 + \phi(\theta). \quad (120)$$

According to Sec. 4.1, the function  $R(\theta)$  is bounded independently of  $\varepsilon$  for  $\theta$  corresponding to  $t$  in an

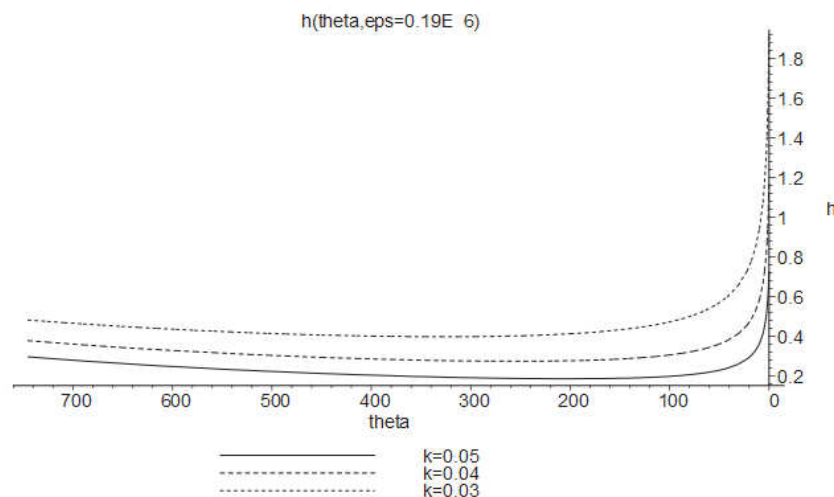


Fig. 3. The function  $h(\theta)$  for relevant values of the parameter  $\kappa$ .

<sup>14</sup>The quantity  $\varepsilon^{3/8}$  measures the width of the boundary layer; it corresponds to  $\varepsilon = 0.187 \times 10^{-6}$ .

interval  $(-\pi/2, -\tau_0 \varepsilon^{3/8})$ . Due to the smallness of  $h(\theta)$  and  $g(\theta)$ , we expect both  $R(\theta)$  and  $\phi(\theta)$  to have a slow variation compared to  $\theta$ ; one might be tempted to replace Eqs. (119a), (119b) with their averages with respect to  $\theta$  (or with respect to  $z$ ). The fact that  $h(\theta)$  is not integrable forbids however this simple averaging.<sup>15</sup> Following Bogolyubov and Mitropolski [1961], we perform a transformation of the dependent variables to new variables by:

$$R_1(\theta) = R(\theta) - \frac{1}{4}h(\theta)R(\theta)^2 \left( \cos z + \frac{1}{3} \cos(3z) \right) \quad (121a)$$

$$\phi_1(\theta) = \phi(\theta) + \frac{1}{4}h(\theta)R(\theta) \left( 3 \sin z + \frac{1}{3} \sin(3z) \right). \quad (121b)$$

The Jacobian of this transformation is  $1 + O(h)$  so that the transformation is *locally* invertible if  $\varepsilon$  and thus  $h(\theta)$ , are small enough. From the boundedness of  $R(\theta)$  established in Sec. 4.1, it follows that  $R_1(\theta)$  is also bounded, independently of  $\varepsilon$ . Moreover, one can show that the transformation (121a), (121b) is in fact invertible at fixed  $\theta$  on its domain of values in the  $(R, \phi)$  plane if  $h(\theta)$  is sufficiently small. This question is discussed in Appendix B. The change of variables (121a), (121b) “removes” the terms of  $O(h)$  in Eqs. (119a), (119b): the equations for  $R_1(\theta)$ ,  $\phi_1(\theta)$  read:

$$\begin{aligned} \frac{dR_1(\theta)}{d\theta} = & \frac{1}{2}R(\theta)g(\theta) \sin(2z) \\ & + R(\theta)^3 h(\theta)^2 \left[ \frac{19}{96} \sin(2z) + \frac{1}{12} \sin(4z) \right. \\ & \left. - \frac{1}{96} \sin(6z) \right] + O\left(h^3, hg, \frac{dh}{d\theta}\right) \end{aligned} \quad (122a)$$

$$\begin{aligned} \frac{d\phi_1(\theta)}{d\theta} = & \frac{1}{2}g(\theta)(1 + \cos(2z)) \\ & + R(\theta)^2 h(\theta)^2 \left[ -\frac{7}{24} - \frac{5}{16} \cos(2z) \right. \\ & \left. - \frac{1}{24} \cos(4z) - \frac{1}{48} \cos(6z) \right] \\ & + O\left(h^3, hg, \frac{dh}{d\theta}\right). \end{aligned} \quad (122b)$$

In these equations, it is understood that  $R(\theta)$ ,  $\phi(\theta)$  are replaced by functions of  $R_1(\theta)$ ,  $\phi_1(\theta)$  obtained by the inversion of Eqs. (121a), (121b). To first order in  $h(\theta)$  the latter reads:

$$R(\theta) = R_1(\theta) + \frac{1}{4}h(\theta)R_1(\theta)^2 \left( \cos z_1 + \frac{1}{3} \cos(3z_1) \right) + O(h^2) \quad (123a)$$

$$\phi(\theta) = \phi_1(\theta) - \frac{1}{4}h(\theta)R_1(\theta) \left( 3 \sin z_1 + \frac{1}{3} \sin(3z_1) \right) + O(h^2) \quad (123b)$$

with

$$z_1 = \theta - \theta_0 + \phi_1(\theta). \quad (124)$$

Since  $h(\theta)^2 \approx \varepsilon^{2\kappa\pi}(\theta_0/\theta)^{3/4}$  is again not integrable, one cannot directly draw conclusions about the behavior of  $R_1(\theta)$  and  $\phi_1(\theta)$  over large intervals of  $\theta$ . An attempt to remove the terms in  $h(\theta)^2$  can achieve this only partially: the equation for  $\phi_1(\theta)$  contains “secular” terms of orders  $h(\theta)^2$  and  $g(\theta)$ , i.e. terms which have nonzero average and which cannot be removed by a further transformation. It is relevant to notice that, in order to remove terms of higher order in Eqs. (122a), (122b), one does not need to resort to the explicit inversion, Eqs. (123a), (123b), but regard  $R(R_1, \phi_1)$ ,  $\phi(R_1, \phi_1)$  as known functions of  $\theta$ , whose derivatives are given by (119a), (119b). With this, we introduce new variables  $R_2, \phi_2$  by:

$$R_2 = R_1 + R(\theta)^3 h(\theta)^2 \left( \frac{19}{192} \cos(2z) + \frac{1}{48} \cos(4z) - \frac{1}{192} \cos(6z) \right) \quad (125a)$$

$$\begin{aligned} \phi_2 = \phi_1 + R(\theta)^2 h(\theta)^2 \left( \frac{5}{32} \sin(2z) + \frac{1}{96} \sin(4z) \right. \\ \left. + \frac{1}{288} \sin(6z) \right). \end{aligned} \quad (125b)$$

As follows from Appendix B, this transformation is invertible under the same conditions as those of Eqs. (121a), (121b). Using (123a), (123b) for  $R(\theta), \phi(\theta)$ , it is true that:

$$R_1 = R_2 + O(h^2) \quad \phi_1 = \phi_2 + O(h^2). \quad (126)$$

<sup>15</sup>As will be apparent, the corrections may not be finite.



Use of (119a), (119b) for  $dR/d\theta$ ,  $d\phi/d\theta$  and of (122a), (122b) for  $dR_1/d\theta$ ,  $d\phi_1/d\theta$  leads to:

$$\begin{aligned} \frac{dR_2(\theta)}{d\theta} &= \frac{1}{2}g(\theta)R(\theta)\sin(2z) \\ &+ O\left(h^3, \frac{dh}{d\theta}, gh\right) \end{aligned} \quad (127a)$$

$$\begin{aligned} \frac{d\phi_2(\theta)}{d\theta} &= -\frac{7}{24}R(\theta)^2h(\theta)^2 + \frac{g(\theta)}{2} \\ &+ \frac{1}{2}g(\theta)\cos(2z) + O\left(h^3, \frac{dh}{d\theta}, gh\right). \end{aligned} \quad (127b)$$

It is convenient to perform a further transformation:

$$\begin{aligned} R_3 &= R_2 + \frac{1}{4}gR\cos(2z) \\ \phi_3 &= \phi_2 - \frac{1}{4}g\sin(2z) \end{aligned} \quad (128)$$

which removes the “nonsecular” terms in  $g(\theta)$  in (127a), (127b). This transformation brings additional terms in the equations corresponding to (127a), (127b), proportional to  $dg/d\theta$  and  $g(\theta)^2$ . The former is dominant and, when its absolute value is integrated, leads to a term of  $O(g(\theta))$ . Now

$$\int_{\theta(-\pi/2)}^{\theta(-\tau_0\varepsilon^{3=8})} h(\theta)^3 d\theta = O\left(\frac{\varepsilon^{3\kappa\pi/2+1/16}}{\ln\left(\frac{1}{\varepsilon}\right)}\right)$$

and vanishes as  $\varepsilon \rightarrow 0$ , so that we conclude from Eqs. (127a)–(128) that:

$$\begin{aligned} R_3(\theta) &= R_{30} + O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta')^3 d\theta'\right) \\ &+ O(g(\theta)) \end{aligned} \quad (129a)$$

$$\begin{aligned} \phi_3(\theta) &= \phi_{30} - \frac{7}{24}\int_{\theta(-\pi/2)}^{\theta} R(\theta')^2 h(\theta')^2 d\theta' \\ &+ \frac{1}{2}\int_{\theta(-\pi/2)}^{\theta} g(\theta') d\theta' \\ &+ O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta')^3 d\theta'\right) \\ &+ O(g(\theta)) \end{aligned} \quad (129b)$$

where  $R_{30}$ ,  $\phi_{30}$  are transformations of the initial conditions at  $t = -\pi/2$  in (116). If, recalling (123a), (123b), (126) we invert the transformations (125a),

(125b) and (121a), (121b) we obtain, using the notations in (116):

$$\begin{aligned} R(\theta) &= \Lambda + O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta')^3 d\theta'\right) \\ &+ O(g(\theta)) + O(h(\theta)) \end{aligned} \quad (130a)$$

$$\begin{aligned} \phi(\theta) &= \Psi_0 - \frac{7}{24}\int_{\theta(-\pi/2)}^{\theta} R_{20}^2 h(\theta')^2 d\theta' \\ &+ \frac{1}{2}\int_{\theta(-\pi/2)}^{\theta} g(\theta') d\theta' \\ &+ O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta')^3 d\theta'\right) \\ &+ O(g(\theta)) + O(h(\theta)). \end{aligned} \quad (130b)$$

In Eq. (130a) we see that, up to possible oscillations of  $O(h(\theta))$ ,  $R(\theta)$  stays constant at its initial value at  $t = -\pi/2$  down to  $t = O(\varepsilon^{3/8-\delta})$ , where the second term containing  $g(\theta)$  may become relevant. Evaluation of the integral in the first term of (130b) leads to:

$$\begin{aligned} \Phi_L &\equiv -\frac{7}{24}\int_{\theta(-\pi/2)}^{\theta(-\tau_0\varepsilon^{3=8})} R_{20}^2 h(\theta)^2 d\theta \\ &= O\left(\frac{\varepsilon^{\kappa\pi}}{\gamma^{1/3}\kappa^{2/3}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{2/3}}\right). \end{aligned} \quad (131)$$

As it will become apparent, bifurcations occur when  $\varepsilon^{\kappa\pi}/\gamma^{1/3} = O(1)$  so that this contribution to the phase due to nonlinear terms (these contain  $h(\theta)$ ) is important: it decays indeed like  $1/(\ln(1/\varepsilon))^{2/3}$ , but this is very slow. The bounded function of  $\kappa$  and  $\varepsilon$  which multiplies the term under the  $O()$  sign has a nontrivial behavior and is shown in Fig. 4; the magnitude of the phase variation implied by (130b) depends on the value of  $\kappa$ , i.e. the damping: at  $\varepsilon^{3/8} = 0.003$ , for  $\kappa = 0.04$  and  $\Lambda = 1$  it is 0.46 rad, but at  $\kappa = 0.02$  it measures 2.01 rad. The second term in (130b) is related to the linear part of (90) and brings a constant contribution at  $t = -\tau_0\varepsilon^{3/8}$ . We conclude this discussion by stating:

**Lemma 5.1.** *The solution of (118a), (118b) with the initial conditions  $R(-\pi/2) = \Lambda$ ,  $\phi(-\pi/2) = \Psi_0$  is given by Eqs. (130a), (130b) for  $t$  in  $(-\pi/2, -\tau_0\varepsilon^{3/8})$ .*

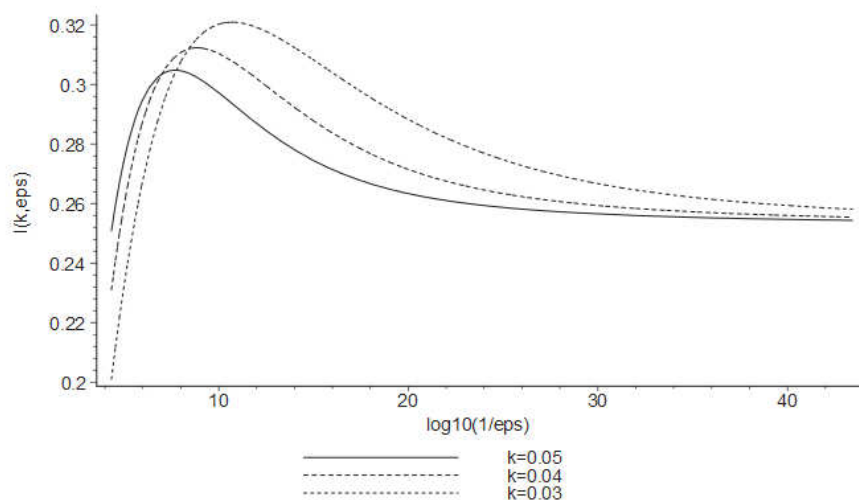


Fig. 4. The dependence on  $\varepsilon$  of the factor multiplying the  $O()$  term in (131).

In Eq. (130a), (130b), the terms containing  $g(\theta)$  originate in the linear part of Eq. (90): so we expect that if we solve the linear part of (90) (for a function  $\tilde{w}_L(\theta)$ ):

$$\frac{d^2 \tilde{w}_L(\theta)}{d\theta^2} + \tilde{w}_L(\theta)(1 + g(\theta)) = 0 \quad (132)$$

with the initial conditions:

$$\begin{aligned} \tilde{w}_L\left(\theta\left(-\frac{\pi}{2}\right)\right) &= \Lambda \cos(\Psi_0 + \Phi_L), \\ \frac{d\tilde{w}_L}{d\theta}\left(\theta\left(-\frac{\pi}{2}\right)\right) &= -\Lambda \sin(\Psi_0 + \Phi_L) \end{aligned} \quad (133)$$

with  $\Phi_L$  of (131) we obtain for  $t$  close to  $-\tau_0 \varepsilon^{3/8}$  values which differ in  $O(h)$  from those of the complete equation. In other words, the only effect of the nonlinear terms in (90) is the addition of a phase  $\Phi_L$  to the simple harmonic evolution. For clarity, we do *not* expect that the solution of (132) with condition (133) approximates the solution of (90) over the *whole* interval  $(-\pi/2, -\tau_0 \varepsilon^{3/8})$ ; only the values near  $t = -\tau_0 \varepsilon^{3/8}$  are presumably well approximated. To render this precise, we analyze (132) using the same transformations (118a), (118b) to new variables — called  $\tilde{R}, \tilde{\phi}$  — which obey equations like (119a), (119b) with  $h(\theta) = 0$ . Performing the change of variables (128) leads in analogy to (130a), (130b) to:

$$\begin{aligned} \tilde{R}_1(\theta) &= \Lambda + O(g(\theta)), \\ \tilde{\phi}_1(\theta) &= \Psi_0 + \Phi_L + \frac{1}{2} \int_{\theta(-\pi/2)}^{\theta} g(\theta') d\theta' \\ &\quad + O(g(\theta)). \end{aligned} \quad (134)$$

Comparison with (130a), (130b) shows that:

$$\begin{aligned} |R(\theta) - \tilde{R}(\theta)| &= O(g(\theta)) + O(h(\theta)) \\ &\quad + O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta')^3 d\theta'\right) \end{aligned} \quad (135a)$$

$$\begin{aligned} |\phi(\theta) - \tilde{\phi}(\theta)| &= O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta')^3 d\theta'\right) \\ &\quad + O(g(\theta)) + O(h(\theta)) \\ &\quad + \frac{7}{24} \int_{\theta}^{-\theta(\tau_0 \varepsilon^{3/8})} \Lambda^2 h(\theta')^2 d\theta'. \end{aligned} \quad (135b)$$

If  $\theta$  is sufficiently close to  $\theta_0 \equiv \theta(-\tau_0 \varepsilon^{3/8})$ , e.g. corresponding to  $t = -\varepsilon^{3/8-\delta}$  for  $\delta$  sufficiently small, but nonzero, then all terms on the right-hand side behave like positive powers of  $\varepsilon$ . As is easily verified, using (117) and (92), these are (up to logarithmic terms) in (135b):  $\varepsilon^{3\kappa\pi/2+1/16}$ ,  $\varepsilon^{8\delta/3}$ ,  $\varepsilon^{\kappa\pi+\delta/2}$ ,  $\varepsilon^{2\kappa\pi-\delta/3}$ , in turn. There is some freedom in the choice of  $\delta$ : a simple choice is  $\delta = 3\kappa\pi/8$  where the  $O(g(\theta))$  term is dominant and the differences (135a), (135b) are of  $O(\varepsilon^{\kappa\pi})$ . If  $\kappa$  is large, this choice becomes inappropriate: if e.g.  $\kappa\pi > 1$ , the  $\tau$ -time interval of  $O(\varepsilon^{-\delta})$  is with this choice larger than the damping time  $1/\gamma$ . We are interested in  $\tau$ -intervals slightly larger than the  $O(1)$  scale but much smaller than  $\varepsilon^{-3/8}$ . We write (with some arbitrariness)

$$\delta = \frac{3r}{8} \quad r = \min\left(\kappa\pi, \frac{1}{2}\right). \quad (136)$$



With this, for  $|\tau(\theta)| < \varepsilon^{-\delta}$ :

$$\begin{aligned} & |w_L(\theta) - \tilde{w}_L(\theta)|, \\ & \left| \frac{dw_L(\theta)}{d\theta} - \frac{d\tilde{w}_L(\theta)}{d\theta} \right| = O(\varepsilon^r). \end{aligned} \quad (137)$$

Now, the variational equation around the reference solution  $X_L(t)$  to the Duffing equation (41) (or (8)) reads:

$$\varepsilon \frac{d^2 \tilde{v}_L}{dt^2} + 2\mu \frac{d\tilde{v}_L}{dt} + 3X_L(t)^2 \tilde{v}_L = 0. \quad (138)$$

With the same Liouville transformation (89) we may write for its solution:

$$\tilde{v}_L(t) = \varepsilon^{\kappa\pi/2+3/16} \frac{\tilde{w}_L(t)\varepsilon^{\kappa(t+\pi/2)}}{(-X_L(t))^{1/2}} \quad (139)$$

where  $\tilde{w}_L(t(\theta))$  obeys (132) with initial conditions (133). Since  $X_L(t) \approx -t^{1/3}$ , one verifies from (139) that, for  $t \approx \varepsilon^{3/8-\delta}$ , both  $v_L(t)$  and  $\tilde{v}_L(t)$  are of  $O(\varepsilon^{\kappa\pi+1/8+\delta/6})$ . It is convenient to revert to the “time”  $\tau$  and to variables  $u_L(\tau)$  and  $\tilde{u}_L(\tau)$  used in the boundary layer region with the scaling (18):

$$\begin{aligned} u_L(\tau) &\equiv \varepsilon^{-1/8} v(t), \\ \tilde{u}_L(\tau) &\equiv \varepsilon^{-1/8} \tilde{v}_L(t), \end{aligned} \quad t = \varepsilon^{3/8} \tau. \quad (140)$$

Since for small  $t$ ,  $\theta \approx -\tau^{4/3}$  both  $du_L/d\tau$ ,  $d\tilde{u}_L/d\tau$  are of  $O(\varepsilon^{\kappa\pi-\delta/6})$  at  $t = -\varepsilon^{3/8-\delta}$ . Using (137), we conclude that, for such values of  $t$  ( $\tau = \varepsilon^{-\delta}$ ):

$$\begin{aligned} & |u_L(\tau) - \tilde{u}_L(\tau)| = O(\varepsilon^{\kappa\pi+r+\delta/6}), \\ & \left| \frac{du_L(\tau)}{d\tau} - \frac{d\tilde{u}_L(\tau)}{d\tau} \right| = O(\varepsilon^{\kappa\pi+r-\delta/6}). \end{aligned} \quad (141)$$

The above is summarized in the statement of

**Lemma 5.2.** *The solutions  $v_L$ , Eq. (89), of Eq. (90) with the initial conditions (116) are approximated together with their derivatives at  $t = -\varepsilon^{3/8-\delta}$  by the solutions of the variational equation (138) with the initial conditions:*

$$\begin{aligned} \tilde{v}_L\left(-\frac{\pi}{2}\right) &= \varepsilon^{3/16+\kappa\pi/2} \Lambda \cos(\Psi_0 + \Phi_L) \\ \frac{d\tilde{v}_L}{d\theta}\left(-\frac{\pi}{2}\right) &= -\varepsilon^{3/16+\kappa\pi/2} \Lambda \sin(\Psi_0 + \Phi_L) \end{aligned} \quad (142)$$

and  $\Phi_L$  of (131) according to the estimates (141) with the definitions (140), (136).

From Eq. (139) one sees that (142) is only a reformulation of (133).

Using the estimates (134) and neglecting the contribution of the integral over  $g(\theta)$  we can write uniform approximations  $\tilde{v}_{c,s}^W(\theta)$  to two special solutions  $\tilde{v}_{c,s}(\theta)$  of the variational equation (138) on the interval  $(-\pi/2, -\varepsilon^{3/8-\delta})$ :

$$\tilde{v}_{c/s}^W(\theta) = \frac{\varepsilon^{\kappa(t+\pi/2)}}{(-X_L(t))^{1/2}} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left( \theta - \theta\left(-\frac{\pi}{2}\right) \right). \quad (143)$$

These solutions (and their approximants) obey:  $\tilde{v}_c(\theta(-\pi/2)) = 1$ ,  $\tilde{v}_s(\theta(-\pi/2)) = 0$ ,  $d\tilde{v}_c/d\theta(\theta(-\pi/2)) = O(\varepsilon \ln(1/\varepsilon))$ ,  $d\tilde{v}_s/d\theta(\theta(-\pi/2)) = 1$ . Comparing with (84) one recognizes in (143) the “WKB approximations” to solutions of Eq. (138). At  $\tau = -\varepsilon^{-\delta}$ ,  $|X_L(t)| \approx \varepsilon^{1/8}\tau^{1/3} = \varepsilon^{1/8-\delta/3}$  and Eq. (134) shows that:

$$\begin{aligned} & |(\tilde{v}_{c,s} - \tilde{v}_{c,s}^W)(\tau = \varepsilon^{-\delta})| \\ &= \varepsilon^{\kappa\pi/2+\delta/6-1/16} \times O\left(\int_{\theta(-\pi/2)}^{\varepsilon} g(\theta) d\theta\right) \\ &= O(\varepsilon^{\kappa\pi/2+r/2+\delta/6-1/16}) \end{aligned} \quad (144)$$

where we used  $g(\theta) \approx 1/\theta^2$  (cf. (92)) and (136). The same quality of approximation holds for  $d\tilde{v}_{c,s}/d\theta$ , but for  $d\tilde{v}_{c,s}/d\tau$  we obtain, in view of  $d\tau/d\theta \approx \tau^{-1/3} = \varepsilon^{\delta/3}$ :

$$\left| \frac{d\tilde{v}_{c,s}}{d\tau} - \frac{d\tilde{v}_{c,s}^W}{d\tau} \right| = O(\varepsilon^{\kappa\pi/2+r/2-\delta/6-1/16}). \quad (145)$$

With the help of (143) a uniform approximation on  $[-\pi/2, -\varepsilon^{-\delta}]$  of the solution of (138) with the initial conditions (142) reads:

$$\begin{aligned} \tilde{v}_L(t) &\approx \tilde{v}_L^W(t) \equiv \Lambda \varepsilon^{3/16+\kappa\pi/2} (\tilde{v}_c^W(\theta) \cos(\Psi_0 + \Phi_L) \\ &\quad - \tilde{v}_s^W(\theta) \sin(\Psi_0 + \Phi_L)) \\ &= \frac{\Lambda \varepsilon^{3/16+\kappa\pi/2}}{(-X_L(t))^{1/2}} \varepsilon^{\kappa\pi/2+t} \\ &\quad \times \cos\left(\theta(t) - \theta\left(-\frac{\pi}{2}\right) + \Psi_0 + \Phi_L\right). \end{aligned} \quad (146)$$

With (145), (146) it follows that, at  $\tau = -\varepsilon^{-\delta}$ :

$$\begin{aligned} & |(\tilde{v}_L - \tilde{v}_L^W)(\tau)| = O(\varepsilon^{1/8+\kappa\pi+r/2+\delta/6}) \\ & \left| \left( \frac{d\tilde{v}_L}{d\tau} - \frac{d\tilde{v}_L^W}{d\tau} \right)(\tau) \right| = O(\varepsilon^{1/8+\kappa\pi+r/2-\delta/6}) \end{aligned} \quad (147)$$

### 5.3. The interval $(-\varepsilon^{3/8-\delta}; 0)$

We notice that, while the “natural” order of magnitude for the boundary layer is  $\varepsilon^{1/8}$  (cf. Eq. (18)), i.e.  $u_L(\tau) = O(1)$ , it turns out that, in fact, with the initial conditions (116),  $u_L(\tau)$  is  $O(\varepsilon^{\kappa\pi})$ , i.e. for small enough  $\varepsilon$  — as argued in Sec. 4.2 — it becomes smaller than the discontinuity of the reference solutions at  $t = 0$ . We compare now the evolution in the interval  $(-\varepsilon^{3/8-\delta}, 0)$  of the solutions of the boundary layer equation with those of the variational equation around the *reference* solution  $X_L(t)$ , with the initial conditions (142). We introduce to this end “macroscopic” variables:

$$U(\tau) = \varepsilon^{-\kappa\pi} u_L(\tau), \quad \tilde{U}(\tau) = \varepsilon^{-\kappa\pi} \tilde{u}_L(\tau) \quad (148)$$

$U(\tau)$  satisfies:

$$\begin{aligned} \frac{d^2 U}{d\tau^2} + 2\gamma \frac{dU}{d\tau} + 3\eta_L(\tau)^2 U \\ + 3\eta_L(\tau)\varepsilon^{\kappa\pi} U^2 + \varepsilon^{2\kappa\pi} U^3 = 0 \end{aligned} \quad (149)$$

and  $\tilde{U}(\tau)$  is a solution of the linear part of (149). In (149),  $\eta_L(\tau) = X_L(t)/\varepsilon^{1/8}$  and  $\gamma$  is given in (18). The solutions of (149) obeying initial conditions at  $\tau = -\varepsilon^{-\delta}$  may be estimated by transforming (149) to an integral equation with the help of two independent solutions  $U_1(\tau)$ ,  $U_2(\tau)$  of the linear part. Using the method of “the variation of the parameters”, we obtain:

$$\begin{aligned} U(\tau) = A_1 U_1(\tau) + A_2 U_2(\tau) \\ + \int_{-\varepsilon}^{\tau} \frac{U_1(\tau)U_2(\tau') - U_2(\tau)U_1(\tau')}{W(U_1, U_2)} \\ \times \varepsilon^{\kappa\pi} (U^2 + \varepsilon^{\kappa\pi} U^3) d\tau' \end{aligned} \quad (150)$$

where  $A_1, A_2$  are such that the values of  $U$  and  $dU/d\tau$  at  $\tau = -\varepsilon^{-\delta}$  are reproduced and  $W(U_1, U_2)$  is the wronskian of  $U_1$  and  $U_2$ . To discuss this equation, we need to know something about the solutions of:

$$\frac{d^2 U}{d\tau^2} + 2\gamma \frac{dU}{d\tau} + 3\eta_L(\tau)^2 U = 0 \quad (151)$$

for  $\tau$  in  $(-\varepsilon^{-\delta}, 0)$ . Writing:

$$U(\tau) = \exp(-\gamma\tau)V(\tau) \quad (152)$$

the equation for  $V(\tau)$  is:

$$\frac{d^2 V}{d\tau^2} + (3\eta_L(\tau)^2 - \gamma^2)V = 0. \quad (153)$$

In Appendix C, we show that we can choose two solutions of (153), called  $V_{c,s}(\tau, \varepsilon)$ , which for large  $-\tau$  assume the “WKB” forms:

$$V_{c,s}^{(as)}(\tau, \varepsilon) = \frac{3^{1/4}}{\Xi(\tau, \varepsilon)^{1/4}} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left( \int_{\tau}^{\tau_a} \Xi(\tau, \varepsilon)^{1/2} d\tau \right) \quad (154)$$

where

$$\Xi(\tau) = 3\eta_L(\tau)^2 - \gamma^2 \approx 3\eta_L(\tau, \varepsilon)^2 \approx 3\tau^{2/3} \quad (155)$$

for  $\tau$  large and  $\tau_a$  is an arbitrary (finite,  $\varepsilon$ -independent) value<sup>16</sup> of  $\tau$ . These solutions may be extended down to  $\tau = 0$  and have a well defined limit as  $\varepsilon \rightarrow 0$ . Indeed, in (153) and (155)  $\eta_L(\tau, \varepsilon)$  is given by the inner expansion (49) (cf. Eq. (74)). According to (49) and (51) (cf. Eq. (74)):

$$\eta_L(\tau, \varepsilon) = \eta_{00L}(\tau) \left( 1 + O\left(\varepsilon^{3/4}\tau^2, \frac{\gamma}{\tau^{5/3}}\right) \right) \quad (156)$$

so that the limit as  $\varepsilon \rightarrow 0$  of expressions (154) at any finite  $\tau$  is obtained by simply replacing  $\eta_L(\tau, \varepsilon)$  by  $\eta_{00L}(\tau)$ . We can also pass to the limit in Eq. (153) and one expects that its solutions tend to the solutions of its limiting form defined correspondingly by the requirement (154). Appendix C gives (straightforward) arguments for this. The solutions  $V_{c,s}(\tau, \varepsilon, \tau_a)$  defined by (154) multiplied by  $\exp(-\gamma\tau)$  (cf. (152)) are chosen as  $U_{1,2}$  in (150). Clearly, the parameter  $\tau_a$  is arbitrary and should drop out in the final expressions.<sup>17</sup> One verifies that, with this choice:

$$W(U_1, U_2) = \exp(-2\gamma\tau) \approx 1, \quad \tau = O(e^{-\delta}). \quad (157)$$

Denoting then:

$$r(\tau) = U(\tau) - A_1 U_1(\tau) - A_2 U_2(\tau) \quad (158)$$

and using the fact that  $|U_1(\tau)|, |U_2(\tau)| < M$  on  $(-\varepsilon^{-\delta}, 0)$ , for some  $M$ , we verify that the operator given by the integral on the right-hand side of (150) defined on the space of functions  $g(\tau)$  continuous on  $[-\varepsilon^{-\delta}, 0]$  endowed with the  $\sup|(1 + \tau^{1/6})g(\tau)|$  norm maps a ball of radius  $\text{const} \times \varepsilon^{\kappa\pi-\delta/2}$  into itself and is, at least for small enough  $\varepsilon$ , a contraction, so that (150) admits a unique solution there. Thus,

<sup>16</sup>It may be chosen as  $\tau_0$  of (84) but need not.

<sup>17</sup>See Sec. 5.4.



the solution of the complete equation (149) departs from the solution of its linear part with the same initial conditions at  $\tau = -\varepsilon^{-\delta}$  by  $O(\varepsilon^{\kappa\pi-\delta/3})$  on  $[-\varepsilon^{-\delta}, 0]$ . For the derivatives one obtains estimates of  $O(\varepsilon^{\kappa\pi-2\delta/3})$ .

We still have to bound the evolution of the distance between two solutions of the linear equation (151) whose values differ by  $O(\varepsilon^{r+\delta/6})$  and their derivatives by  $O(\varepsilon^{r-\delta/6})$  at  $\tau = -\varepsilon^{-\delta}$  (cf. Eqs. (141) and (136)). This is a direct application of (C.19) in Appendix C, from which one deduces:

$$|\Delta U(\tau = 0)|, \quad \left| \frac{d\Delta U}{d\tau}(\tau = 0) \right| = O(\varepsilon^r). \quad (159)$$

Recalling (148), we may summarize the foregoing by:

**Lemma 5.3.** *If a solution  $v_L(t) = \varepsilon^{1/8}u_L(\tau)$  of the Duffing equation (83) differs from a solution  $\tilde{v}_L(t) = \varepsilon^{1/8}\tilde{u}_L(t)$  of the variational equation around the reference solution  $X_L(t)$  at  $t = \varepsilon^{3/8-\delta}$  according to (141), then at  $t = 0$ :*

$$\begin{aligned} |(u_L - \tilde{u}_L)(\tau = 0)| &= O(\varepsilon^{\kappa\pi+r-\delta/3}), \\ \left| \left( \frac{du_L}{d\tau} - \frac{d\tilde{u}_L}{d\tau} \right)(\tau = 0) \right| &= O(\varepsilon^{\kappa\pi+r-2\delta/3}) \end{aligned} \quad (160)$$

(recall  $\delta = 3r/8$ ).

It is useful to recall the “constitution” of the exponents of  $\varepsilon$  in (160) which measure the order of magnitude of the approximation: a factor  $\varepsilon^{\kappa\pi/2}$  originates in the initial conditions (142); a factor  $\varepsilon^{\kappa\pi/2}$  is a result of the damping; these two factors control the magnitude of both  $u_L(\tau)$  and  $\tilde{u}_L(\tau)$  in (160); the remaining factor  $\varepsilon^{r-\delta/3}$  (or  $\varepsilon^{r-2\delta/3}$ ) states that the “macroscopic” quantities (148) are close to each other at  $\tau = -\varepsilon^{-\delta}$  as stated in (141) and this distance may increase a little as we move from  $\tau = -\varepsilon^{-\delta}$  to  $\tau = 0$ .

#### 5.4. Summary of the quarter period map for $\varepsilon < 0$

The conclusion in Sec. 5.3 is that to  $O(\varepsilon^{\kappa\pi+r-2\delta/3})$  the Poincaré map  $\mathbb{P}_L : (v_L, dv_L/d\theta(\pi/2)) \Rightarrow (u_L, du_L/d\tau)(\tau = 0)$  is given by the solutions of the variational equation around the reference solution  $X_L(t)$  with initial conditions (142), where  $(\Lambda, \Psi_0)$  are related to  $(v_L, dv_L/d\theta(t = -\pi/2))$  by (116) and  $\Xi_L$  is given by (155). We can write a more explicit

form of  $\mathbb{P}_L$  using the combinations:

$$\begin{aligned} \tilde{V}_c &= \varepsilon^{\kappa\pi/2-1/16}(V_c \cos(\Omega(\tau_a)) - V_s \sin(\Omega(\tau_a))) \\ \tilde{V}_s &= \varepsilon^{\kappa\pi/2-1/16}(V_c \sin(\Omega(\tau_a)) + V_s \cos(\Omega(\tau_a))) \end{aligned} \quad (161)$$

with

$$\begin{aligned} \Omega(\tau_a) &= \theta(\tau_a) - \theta\left(-\frac{\pi}{2}\right) \\ &\approx \int_{-\pi/(2\varepsilon^{3/8})}^{\tau_a} \Xi(\tau)^{1/2} d\tau. \end{aligned} \quad (162)$$

They are chosen so that  $\varepsilon^{1/8}\tilde{V}_{c,s}$  are equal at large  $\tau = -\varepsilon^{-\delta}$  to the WKB approximation (143) to the solutions  $\tilde{v}_{c,s}$  of the variational equation (138), defined by initial conditions at  $t = -\pi/2$  (see text following Eq. (143)). The combinations (161) are, when multiplied by  $\exp(-\gamma\tau)$ , exact solutions of (151). The expression:

$$\begin{aligned} \tilde{v}_{aL}(t) &\equiv \Lambda \varepsilon^{3/16+\kappa\pi/2}(\tilde{V}_c \cos(\Psi_0 + \Phi_L) \\ &\quad - \tilde{V}_s \sin(\Psi_0 + \Phi_L)) \\ &= \Lambda \varepsilon^{\kappa\pi+1/8}(V_c \cos(\Omega(\tau_a) + \Psi_0 + \Phi_L) \\ &\quad - V_s \sin(\Omega(\tau_a) + \Psi_0 + \Phi_L)) \\ &\equiv \varepsilon^{1/8}\tilde{u}_{aL}(t) \end{aligned} \quad (163)$$

is a solution of the variational equation (138) which differs at  $\tau = -\varepsilon^{-\delta}$  from the solution  $\tilde{v}_L(t)$  of Lemma 5.2 as described in (147). With the same argument used in Eq. (159) and in Appendix C, this difference propagates down to  $\tau = 0$ : recalling the definitions of  $\tilde{u}_L(t)$  (cf. Lemma 5.3 and (160)) and  $\tilde{u}_{aL}(t)$  (cf. Eq. (163)) then

$$\begin{aligned} |(\tilde{u}_L - \tilde{u}_{aL})(\tau = 0)|, \\ \left| \left( \frac{d\tilde{u}_L}{d\tau} - \frac{d\tilde{u}_{aL}}{d\tau} \right)(\tau = 0) \right| &= O(\varepsilon^{\kappa\pi+r/2}). \end{aligned} \quad (164)$$

Thus, we can use the solution  $\tilde{u}_{aL}(t)$ , which involves the functions  $V_{c,s}$  defined by (154) and having a well defined limit as  $\varepsilon \rightarrow 0$  (cf. (156)), to express the left-hand side Poincaré mapping  $\mathbb{P}_L$  in a simpler form: taking (160) and (164) into account:

$$\begin{aligned} |u_L(\tau) - \tilde{u}_{aL}(\tau)| \\ < |u_L(\tau) - \tilde{u}_L(\tau)| + |\tilde{u}_L(\tau) - \tilde{u}_{aL}(\tau)| \\ &= O(\varepsilon^{\kappa\pi+r-\delta/3}, \varepsilon^{\kappa\pi+r/2}) \\ &= O(\varepsilon^{\kappa\pi+r/2}) \end{aligned} \quad (165)$$

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and a similar estimate for the derivative; we can then state:

**Theorem 5.1.** *The quarter period Poincaré map  $\mathbb{P}_L : (\Lambda, \Psi) \Rightarrow (u_L, du_L/d\tau(\tau = 0))$  is given by:*

$$\begin{aligned} u_L(0) = & \Lambda \varepsilon^{\kappa\pi} (V_c(0, \varepsilon, \tau_a) \cos(\Psi + \Phi_L + \Omega(\tau_a)) \\ & - V_s(0, \varepsilon, \tau_a) \sin(\Psi + \Phi_L + \Omega(\tau_a)) \\ & + O(\varepsilon^{r/2})) \end{aligned} \quad (166a)$$

$$\begin{aligned} \frac{du_L}{d\tau}(0) = & \Lambda \varepsilon^{\kappa\pi} \left( \frac{dV_c}{d\tau}(0, \varepsilon, \tau_a) \cos(\Psi + \Phi_L + \Omega(\tau_a)) \right. \\ & - \frac{dV_s}{d\tau}(0, \varepsilon, \tau_a) \sin(\Psi + \Phi_L + \Omega(\tau_a)) \\ & \left. + O(\varepsilon^{r/2}) \right). \end{aligned} \quad (166b)$$

Recalling the definition (131) of  $\Phi_L$ , Eqs. (166a), (166b) show that circles of radius  $\Lambda \varepsilon^{3/16 + \kappa\pi/2}$  in the  $(v_L(-\pi/2), dv_L/d\theta(-\pi/2))$  plane are mapped onto ellipses in the  $(u_L(0), du_L/d\tau(0))$  plane, with a  $\Lambda$ -dependent phase  $\Phi_L$  (see Fig. 5). Thus the disk  $\Lambda < \text{const}$  undergoes under  $\mathbb{P}_L$  a *torsion*. All quantities in (166a), (166b) may be obtained from a numerical solution of the variational equation only in an interval near  $t = 0$ : it is enough to find — for a given choice of  $\tau_a$  — the solutions  $V_{c,s}(\tau, \varepsilon, \tau_a)$

obeying the boundary condition given by (154) at some large  $\tau = \varepsilon^{-\delta}$  — and extend numerically the solution down to  $\tau = 0$ . One may object that the mapping seems to depend on an arbitrary parameter  $\tau_a$ : this is, however, not the case. The reason is that in the form (146) of the solutions at large  $\tau$ , the parameter  $\tau_a$  does not appear at all. It is thus absent in the solutions  $\tilde{V}_{c,s}$  of (161) which match (143) at  $\tau = -\varepsilon^{-\delta}$ . The rotations (161) which depend on  $\tau_a$  leave the sums

$$A^2 \equiv V_c(0, \varepsilon, \tau_a)^2 + V_s(0, \varepsilon, \tau_a)^2,$$

$$B^2 \equiv \left( \frac{dV_c}{d\tau} \right)^2(0, \varepsilon, \tau_a) + \left( \frac{dV_s}{d\tau} \right)^2(0, \varepsilon, \tau_a)$$

invariant. Further, the angle  $\chi$  between the vectors  $(V_c, V_s)$  and  $(dV_c/d\tau, dV_s/d\tau)$  is also invariant. The image of the circle  $\Lambda = \text{const}$  under  $\mathbb{P}_L$  is

$$\frac{u_L^2}{A^2} + \frac{\left( \frac{du_L}{d\tau} \right)^2}{B^2} - 2 \frac{u_L \left( \frac{du_L}{d\tau} \right)}{AB} \cos \chi = \Lambda^2 \sin^2 \chi.$$

It depends only on these three quantities, so that the independence of  $\mathbb{P}_L$  on  $\tau_a$  is apparent. Moreover, the quantities  $V_{c,s}(0, \varepsilon, \tau_a)$  and their derivatives have a limit as  $\varepsilon \rightarrow 0$ , according to the remarks surrounding (156) and to the discussion of Appendix C. In numerical calculations we choose  $\tau_a = -10$ : at this value we can approximate:  $\eta(\tau) \approx \sin(\tau \varepsilon^{3/8})^{1/3} / \varepsilon^{1/8}$ . The limiting values for  $\varepsilon = 0$  of the constants in (166a), (166b) are found to be:

$$V_c(0, 0, \tau_a = -10) \approx -1.163$$

$$\frac{dV_c}{d\tau}(0, 0, \tau_a = -10) \approx -0.178$$

$$V_s(0, 0, \tau_a = -10) \approx -0.0876$$

$$\frac{dV_s}{d\tau}(0, 0, \tau_a = -10) \approx -1.5086$$

(167)

Then:

$$A = 1.1659, \quad B = 1.5191, \quad \chi = 78.960^\circ.$$

## 6. The Right-Hand Side Poincaré Map

### 6.1. The continuation of the reference solution $X_L$ to $t > 0$

As one sees in Fig. 6, the continuation of  $X_L$  to  $t > 0$  traverses first the  $x$ -axis and then approaches

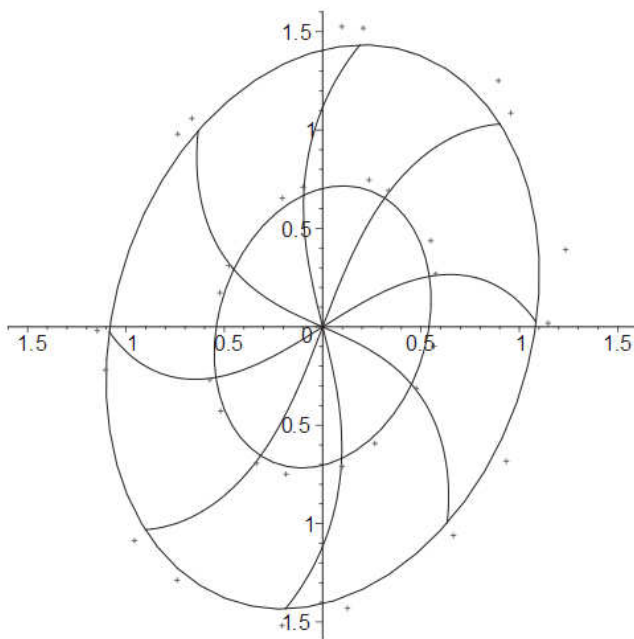


Fig. 5. The image of the disk  $\Lambda = 1$  under  $\mathbb{P}_L$  with the rays  $\Psi = n\pi/4$ .



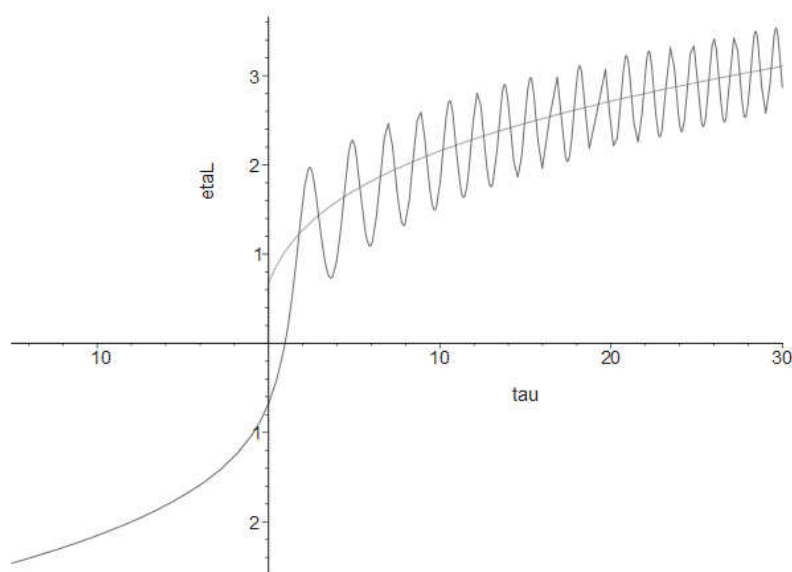


Fig. 6. The continuation of the reference solution  $\eta_L(\tau) = X_L(\tau)/\varepsilon^{1/8}$  to  $t > 0$ .

the reference solution  $X_R$  as  $t$  increases, oscillating around it. We describe in the following, this behavior in more detail.

#### The interval $0 < t < \tau_0 \varepsilon^{3/8}$

The initial values of the solution  $X_L$  at  $\tau = 0$  are  $(\varepsilon^{1/8} \eta_L(0), \varepsilon^{1/8} d\eta_L/d\tau(0)) \approx \varepsilon^{1/8}(-0.677, 0.472)$ . In the limit  $\varepsilon \rightarrow 0$ , the boundary layer equation

$$\frac{d^2 \eta}{d\tau^2} + \eta^3 = \tau \quad (168)$$

admits the symmetry  $\tau \rightarrow -\tau, \eta \rightarrow -\eta$ ; as a consequence, it is true that  $\eta_L(\tau) = -\eta_R(-\tau)$ , where  $\eta_L, \eta_R$  are the solutions behaving like  $-\tau^{1/3}, \tau^{1/3}$  as  $\tau \rightarrow -\infty, \infty$  in turn. Therefore, in this limit,  $\eta_L(0) = -\eta_R(0), d\eta_L/d\tau(0) = d\eta_R/d\tau(0)$  (cf. Sec. 3.3). For  $t > 0$  (and any  $\varepsilon > 0$ ), we denote  $\Delta\eta(\tau, \varepsilon) = \eta_L - \eta_R$ ; it obeys the equation:

$$\begin{aligned} \frac{d^2 \Delta\eta}{d\tau^2} + 2\gamma \frac{d\Delta\eta}{d\tau} + 3\eta_R^2 \Delta\eta \\ + 3\eta_R(\Delta\eta)^2 + (\Delta\eta)^3 = 0. \end{aligned} \quad (169)$$

With the same argument as in Sec. 4.1, Eq. (103) we verify that the energy associated with (169):

$$\begin{aligned} E(\tau) = \frac{1}{2} \left( \frac{d\Delta\eta}{d\tau} \right)^2 + \frac{3}{2} \eta_R^2 (\Delta\eta)^2 \\ + \eta_R (\Delta\eta)^3 + \frac{(\Delta\eta)^4}{4} \end{aligned} \quad (170)$$

obeys (if  $E > 1$ ) the inequality:

$$\frac{dE}{d\tau} \leq \text{const} \frac{d\eta_R}{d\tau} E^{3/4} \quad (171)$$

from which one concludes that  $E(\tau)$  and thus  $\Delta\eta(\tau)$  are bounded at  $t = \tau_0 \varepsilon^{3/8}$ . Clearly, the same is true for all solutions starting in a disk of radius  $\varepsilon^{\kappa\pi}$  (or of any finite radius) around  $(\Delta\eta(0), d\Delta\eta/d\tau(0))$ . Here  $\tau_0$  is a time in the boundary layer region (typically  $\tau_0 = 10$ ).

#### The interval $\tau_0 \varepsilon^{3/8} < t < \pi/2$

As in Secs. 4.1 and 5.1 we write for  $\Delta X(t) \equiv \Delta\eta(t)\varepsilon^{1/8}$ :

$$\Delta X(t) = \frac{\tilde{w}_{L,R}(\theta_R)}{X_R^{1/2}} \varepsilon^{\kappa t + 3/16} \quad (172)$$

with  $\theta_R$  given in Eq. (84). **Note:** In the rest of this section we shall drop the subscript “R” on  $\theta_R$  since it is clear that we confine ourselves to the time interval  $[0, \pi/2]$ . The subscript “L” is appended to quantities related to the continuation of  $t > 0$  of the left-hand reference solution  $X_L(t)$  or to departures from it.

The function  $\tilde{w}_{L,R}$  obeys:

$$\begin{aligned} \frac{d^2 \tilde{w}_{L,R}}{d\theta^2} + \tilde{w}_{L,R}(1 + g(\theta)) + k(\theta) \tilde{w}_{L,R}^2 \\ + \frac{1}{3} k(\theta)^2 \tilde{w}_{L,R}^3 = 0 \end{aligned} \quad (173)$$

where (cf. (96))

$$k(\theta) = \frac{\varepsilon^{3/16+\kappa t}}{X_R^{3/2}} = \frac{\exp(-\gamma\tau)}{\eta_R^{3/2}}, \quad (174)$$

$X_R(t) \equiv \varepsilon^{1/8}\eta_R(t(\theta))$  and  $g(\theta)$  is given in (90) with the interchange  $-X_L \leftrightarrow X_R$ . As in Sec. 5.2, we move over to polar coordinates:

$$\tilde{w}_{L,R} = R_L(\theta) \cos(\theta + \phi_L(\theta)) \quad (175a)$$

$$\frac{d\tilde{w}_{L,R}}{d\theta} = -R_L \sin(\theta + \phi_L(\theta)) \quad (175b)$$

and obtain equations completely analogous to (119a), (119b) with the change  $h \rightarrow -k$ :

$$\begin{aligned} \frac{dR_L}{d\theta} &= \frac{g(\theta)R_L(\theta)}{2} \sin(2z) \\ &+ \frac{k(\theta)R_L^2(\theta)}{4} (\sin z + \sin 3z) \\ &+ \frac{k(\theta)^2 R_L(\theta)^3}{12} \left( \sin 2z + \frac{\sin 4z}{2} \right) \end{aligned} \quad (176a)$$

$$\begin{aligned} \frac{d\phi_L}{d\theta} &= \frac{g(\theta)}{2} (1 + \cos 2z) \\ &+ \frac{1}{4} k(\theta) R_L(\theta) (3 \cos z + \cos 3z) \\ &+ k(\theta)^2 R_L(\theta)^2 \left( \frac{3}{2} + 2 \cos 2z + \frac{1}{2} \cos 4z \right) \end{aligned} \quad (176b)$$

with  $z = \theta + \phi_L(\theta)$ . We imitate now the arguments in Sec. 5 concerning averaging and “remove” first the terms in  $k(\theta)$  in (176a), (176b) by a transformation of the dependent variables similar to Eqs. (121a), (121b). One has to realize that, although the equations are similar to those of the previous section, the function  $k(\theta)$  is of  $O(1)$  when  $t$  is of  $O(\varepsilon^{3/8})$ , in opposition to  $h(\theta)$  which is of  $O(\varepsilon^{\kappa\pi})$  (cf. Eq. (117)). However, it is monotonically decreasing with  $\theta$  and arbitrarily small for large  $\theta$ . According to Appendix B, the inversion of the transformations (121a), (121b) at fixed  $\tau > \tau_0$  is possible if  $\tau_0$  is sufficiently large.

We obtain a set of equations for the functions  $R_{1,L}, \phi_{1,L}$  very similar to Eqs. (122a), (122b). The initial conditions at  $\tau_0$  are different to  $O(1)$  to those for  $R_L(\theta), \phi_L\theta$  in (176a), (176b). The boundedness of the solutions of these equations is not immediately apparent since  $k(\theta)^2 \approx (\theta_0/\theta)^{3/4}$  is not integrable (clearly, the integral over  $k(\theta)^2$  is finite for

finite  $\varepsilon$  but diverges as  $\varepsilon \rightarrow 0$ ). We perform thus a second transformation, similar to (125a), (125b), which separates off a term  $(-7/24)R_L(\theta)^2 k(\theta)^2$  in the equation for  $\phi_{2,L}(\theta)$ .

We denote:

$$\Phi_{L,R}(\theta) \equiv -\frac{7}{24} \int_{\theta_0}^{\theta} R_L(\theta)^2 k(\theta)^2 d\theta. \quad (177)$$

Contrary to the phase  $\Phi_L$  of Eq. (131), whose magnitude depends on the ratio  $\varepsilon^{\kappa\pi}/\gamma^{1/3}$ , the value at  $\theta(\pi/2)$  of the additional phase  $\Phi_{L,R}(\theta)$ , Eq. (177), appearing for  $t > 0$  in the oscillations of the extension of  $X_L(t)$  around  $X_R(t)$  is truly divergent as  $\varepsilon$  vanishes. Using the estimate:

$$\frac{2}{\pi}t \leq \sin t \leq t, \quad 0 < t < \frac{\pi}{2}$$

one verifies that:

$$\begin{aligned} \int_{\tau_0\varepsilon^{3/8}}^{\pi/2} k(\theta)^2 \frac{d\theta}{dt} dt &\approx \frac{1}{\varepsilon^{1/8}\kappa^{1/3} \ln^{1/3}\left(\frac{1}{\varepsilon}\right)} \\ &\times \int_0^\infty \frac{\exp(-2u)du}{u^{2/3}} \\ &= O(\gamma^{-1/3}). \end{aligned} \quad (178)$$

As in Sec. 5.2, it turns out that  $dR_L/d\theta$  is very small for large  $\theta$  and thus we expect  $R_L(\theta)$  to approach a constant value  $R_{L,f} \equiv R_L(t(\theta) = \pi/2)$ . If this value is nonzero, the additional phase  $\Phi_{L,R}$  is indeed divergent as  $\varepsilon \rightarrow 0$ . Now, the function  $X_L(t)$  (and thus  $\eta_L(\theta)$ ) still depends on the value of  $\varepsilon$ ; so do the corresponding values  $R_{L,f}$ , for which we write for clarity  $R_{L,f}(\varepsilon)$ . In Appendix D we show

**Lemma 6.1.** *As  $\varepsilon \rightarrow 0$ , the values  $R_{L,f}(\varepsilon)$  tend to a limit  $R_{L,f}(0)$ . This limit is obtained by solving Eqs. (127a), (127b) with  $-h(\theta)$  replaced by  $k(\theta)$ , Eq. (174) where  $\eta_R(\theta, \varepsilon)$  is replaced by  $\eta_{00R}(\tau(\theta))$  of (64a) and  $\gamma$  is set equal to zero.*

For  $\gamma = 0$  we obtain in (174)  $k(\theta) = 1/\eta_R(\theta)^{3/2} \approx 1/\tau(\theta)^{1/2}$ . A numerical evaluation leads to  $R_f(0) \approx 0.84$ . If we accept this as a “proof” that  $R_f(0) \neq 0$ , we can state

**Lemma 6.2.** *The value of the secular term  $\Phi_{L,R}(\theta)$  at  $\theta(t = \pi/2)$  is  $O(\gamma^{-1/3})$ .*



We define the “rest phase” left after the removal of the secular term as:

$$\bar{\phi}_L(\theta) \equiv \phi_L(\theta) - \Phi_{L,R}(\theta). \quad (179)$$

This “rest phase” is also  $\varepsilon$ -dependent (so that we should write  $\bar{\phi}_{L,\varepsilon}(\theta)$ ), however, in a “harmless” manner: let  $\bar{\phi}_{L,\varepsilon}(t(\theta) = \pi/2) \equiv \bar{\phi}_{L,f}(\varepsilon)$ . With the same argument leading to Lemma 6.1 we show in Appendix D:

**Lemma 6.3.** *As  $\varepsilon \rightarrow 0$ , the values  $\bar{\phi}_{L,f}(\varepsilon)$  tend to a limit  $\bar{\phi}_{L,f}(0)$ . This limit is obtained by solving Eqs. (127a), (127b) with  $-h(\theta)$  changed to  $k(\theta)$  of (174) with the same replacements as in Lemma 6.1.*

To conclude, taking (172) into account, the continuation of  $X_L(t)$  to  $t > 0$  oscillates around the reference solution  $X_R(t)$ ; the departure from  $X_R$  reaches the value  $0.84\varepsilon^{\kappa\pi/2+3/16}$  at  $t = \pi/2$ ; as follows from the definition (84) of the variable  $\theta_R$ , the number of oscillations with frequency proportional to  $X_R(t)$  increases indefinitely as  $\varepsilon \rightarrow 0$  and there is an additional phase, which also increases indefinitely in this limit, as shown by Eq. (178).

## 6.2. The variational equation around $X_L(t)$ for $t > 0$

In Sec. 5, we have seen that all those solutions of Duffing’s equation which start at  $t = -\pi/2$  in a disk of radius  $\Lambda\varepsilon^{\kappa\pi/2+3/16}$  around the left-hand side reference solution  $X_L(t)$  land in a disk of radius  $\varepsilon^{\kappa\pi}$  around the values  $(\eta_L(0), d\eta_L/d\tau(0))$  (cf. Sec. 5.4, Eqs. (166a), (166b); their departure from  $X_L(t) \equiv \varepsilon^{1/8}\eta_L(\tau)$  is denoted there by  $\varepsilon^{1/8}u_L(\tau)$ . For  $t > 0$ , we consider the departures from  $X_R(t) \equiv \varepsilon^{1/8}\eta_R(\tau)$ :

$$\begin{aligned} u_R(\tau) &= u_L(\tau) + \eta_L(\tau) - \eta_R(\tau) \\ &\equiv u_L(\tau) + \Delta\eta(\tau) \end{aligned} \quad (180)$$

with  $\Delta\eta(\tau)$  of (169). In analogy to Eq. (172), we write:

$$u_R(\theta) = \frac{w(\theta)\varepsilon^{\kappa t}}{\eta_R^{1/2}} \quad (181)$$

and define  $R(\theta), \phi(\theta)$  in analogy to Eqs. (175a), (175b):

$$w(\theta) = R \cos(\theta + \phi) \quad (182a)$$

$$\frac{dw}{d\theta} = -R \sin(\theta + \phi). \quad (182b)$$

The right-hand side Poincaré map around  $X_R(t)$  may then be written as

$$\begin{aligned} \mathbb{P}_R : \left( u_R(0), \frac{du_R}{d\tau}(0) \right) \\ \Rightarrow \left( R \left( t(\theta) = \frac{\pi}{2} \right), \phi \left( t(\theta) = \frac{\pi}{2} \right) \right). \end{aligned} \quad (183)$$

Since we are interested only in a small neighborhood of the point  $(\Delta\eta(0), d\Delta\eta/d\tau(0)) \approx (2\eta_L(0), 0)$ , we expand the Poincaré map in a Taylor series around it:  $(\phi_{L,f}(\varepsilon) \equiv \phi_L(t(\theta) = \pi/2, \varepsilon)$ , cf. Eq. (175b))

$$\begin{aligned} \mathbb{P}_R \left( u_R(0), \frac{du_R}{d\tau}(0) \right) \\ = (R_{L,f}(\varepsilon), \phi_{L,f}(\varepsilon)) + \mathbb{D}\mathbb{P}_R \left( \Delta\eta(0), \frac{d\Delta\eta}{d\tau}(0) \right) \\ \times \left( u_L(0), \frac{du_L}{d\tau}(0) \right) + \dots \end{aligned} \quad (184)$$

where  $(u_L(0), du_L/d\tau(0))$  is, according to Eqs. (166a), (166b),  $O(\varepsilon^{\kappa\pi})$ . We write the mapping  $\mathbb{P}_R$  as a composition of three transformations:

$$\mathbb{P}_R = \mathbb{P}_f \circ \mathbf{T} \circ \mathbb{P}_i \quad (185)$$

given by:

$$\mathbb{P}_i : \left( u_R(0), \frac{du_R}{d\tau}(0) \right) \Rightarrow \left( u_R(\tau_0), \frac{du_R}{d\tau}(\tau_0) \right) \quad (186a)$$

$$\mathbf{T} : \left( u_R(\tau_0), \frac{du_R}{d\tau}(\tau_0) \right) \Rightarrow (R(\theta(\tau_0)), \phi(\theta(\tau_0))) \quad (186b)$$

$$\begin{aligned} \mathbb{P}_f : (R(\theta(\tau_0)), \phi(\theta(\tau_0))) \\ \Rightarrow \left( R \left( \theta \left( t = \frac{\pi}{2} \right) \right), \phi \left( \theta \left( t = \frac{\pi}{2} \right) \right) \right) \end{aligned} \quad (186c)$$

where  $\tau_0$  is a “time” in the boundary layer already introduced in Sec. 6.1 following Eq. (171); it is for convenience also chosen as origin of the variable  $\theta_R$ , Eq. (84). Corresponding to the composition (185) we write for the derivative:

$$\begin{aligned} \mathbb{D}\mathbb{P}_R \left( \Delta\eta(0), \frac{d\Delta\eta}{d\tau}(0) \right) \\ = \mathbb{D}\mathbb{P}_f(R_L(\tau_0), \phi_L(\tau_0)) \circ \mathbb{D}\mathbf{T} \left( \Delta\eta(\tau_0), \frac{d\Delta\eta}{d\tau}(\tau_0) \right) \\ \circ \mathbb{D}\mathbb{P}_i \left( \Delta\eta(0), \frac{d\Delta\eta}{d\tau}(0) \right). \end{aligned} \quad (187)$$

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The elements of the Jacobian matrices appearing in (187) are the values of solutions of the variational equation around  $X_L(t)$  for  $t > 0$  at  $\tau_0$  and  $t = \pi/2$  with appropriate initial conditions. We evaluate next these elements.

### The interval $0 < t < \tau_0 \varepsilon^{3/8}$

The variational equation around  $X_L(t)$  reads:

$$\frac{d^2 \delta u}{d\tau^2} + 2\gamma \frac{d\delta u}{d\tau} + 3\eta_L(\tau)^2 \delta u = 0. \quad (188)$$

On the bounded interval  $0 \leq \tau \leq \tau_0$  this equation has bounded solutions; moreover, as  $\varepsilon \rightarrow 0$ , these solutions tend uniformly to those of the equation obtained by letting formally  $\varepsilon = 0$  in (188). This means setting  $\gamma = 0$  in (188) and replacing  $\eta_L(\tau)$  by the continuation to  $t > 0$  of the first term  $\eta_{00L}(\tau)$  in the expansion (74).

The rapid oscillations of  $\eta_L(\tau)$  for  $\tau > 0$  (see Fig. 6) lead to solutions of Eq. (188) with a more complicated behavior than those on the l.h.s.<sup>18</sup> Whereas the image at  $\tau = 0$  of the circle  $\Lambda = 1$  at  $t = -\pi/2$  is a (torsioned) ellipse (see Fig. 5), the deformation of the latter under the flow for  $\tau > 0$  is considerable, see Fig. 7, which shows the image of the (approximate) ellipse at  $\tau = 0$  (crosses) at times  $\tau = 3$  (boxes) and  $\tau = 10$  (diamond) for values of  $\varepsilon^{3/8} = 0.003$ ,  $\kappa = 0.04$ . The origin is now chosen at  $(\eta_R(0), d\eta_R/d\tau(0))$ .

### The transformation $\mathbf{T}$

With the help of Eq. (181) and the definition of the variable  $\theta_R$ , one verifies that the Jacobian of the transformation:

$$\mathbf{T}_1 : \left( u_R, \frac{du_R}{d\tau} \right) (\tau_0) \Rightarrow \left( w, \frac{dw}{d\theta} \right) (\tau_0)$$

is:

$$\det \mathbb{D}\mathbf{T}_1 = \frac{1}{\sqrt{3}} \exp(-\gamma\tau_0).$$

All elements of  $\mathbb{D}\mathbf{T}_1$  are continuous at  $\varepsilon = 0$ . The supplementary transformation<sup>19</sup>  $\mathbf{T}_2 : (w, dw/d\theta) \Rightarrow (R, \phi)$  given by Eqs. (182a), (182b) has a Jacobian equal to  $-1/R(\tau_0)$ ; we assume numerically that  $R(\tau_0) \neq 0$ ; it tends to a nonzero value  $R_0(\tau_0)$  as  $\varepsilon \rightarrow 0$ .

### The interval $\tau_0 \varepsilon^{3/8} < t < \pi/2$

The Jacobian matrix  $\mathbb{D}\mathbf{P}_t$  is more complicated: it contains elements which diverge as  $\varepsilon \rightarrow 0$ ; the reason is that the contribution coming from the variation of the term  $\Phi_{L,R}$  of (177) (see Lemma 6.2) is divergent in this limit. It is thus convenient to study first a “reduced” transformation:

$$\mathbb{P}_{\text{fr}} : (R, \phi)(\tau_0) \Rightarrow (R, \bar{\phi}) \left( t = \frac{\pi}{2} \right) \quad (189)$$

where  $\bar{\phi}$  is the “rest phase” defined for each solution in analogy to (179), by subtraction of the

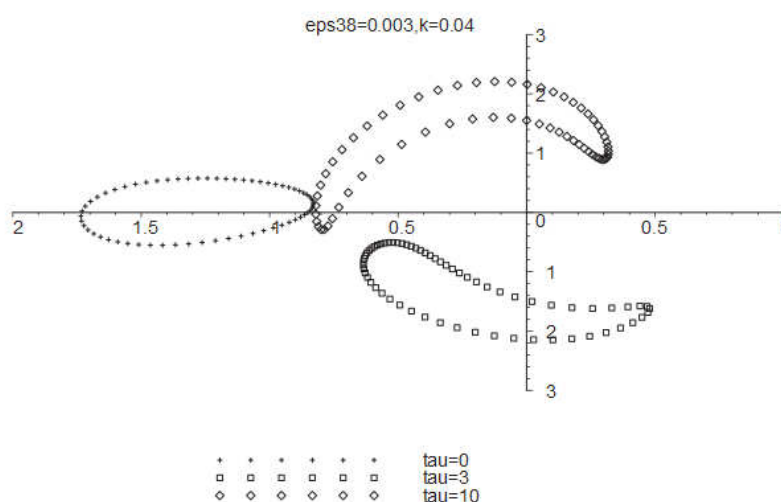


Fig. 7. The deformation of the domain in Fig. 5 at small  $\tau > 0$ .

<sup>18</sup>The corrections to the WKB formulae are determined by the function  $g(\theta)$  (analog of Eq. (92)) which contains the first and second derivatives of the rapidly oscillating  $\eta_L(\tau)$ .

<sup>19</sup> $\mathbf{T} = \mathbf{T}_2 \circ \mathbf{T}_1$ .



“secular” term. To this transformation, we associate the Jacobian matrix  $\mathbb{D}\mathbb{P}_{\text{fr}}$ . Related to it, we show:

**Lemma 6.4.** *The matrix elements of  $\mathbb{D}\mathbb{P}_{\text{fr}}$  are bounded and continuous with respect to  $\varepsilon$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* The proof is similar to the one in Appendix D and is based on a qualitative study of the solutions of the variational equation around the solutions  $R_{4L,0}, \phi_{4L,0}$  of Eqs. (D.4a), (D.4b) and of the solutions  $R_{4L,\varepsilon}, \phi_{4L,\varepsilon}$  of (D.6a), (D.6b), in turn. According to the choice of initial conditions, these solutions, denoted generally by  $\delta R(\theta), \delta \phi(\theta)$  in the following, may be identified with the partial derivatives  $\partial R(\theta)/\partial R_0, \partial \phi(\theta)/\partial R_0$  (if  $\delta R(0) = 1, \delta \phi(0) = 0$ ) or  $\partial R(\theta)/\partial \phi_0, \partial \phi(\theta)/\partial \phi_0$  (if  $\delta R(0) = 0, \delta \phi(0) = 1$ ), where  $R_0, \phi_0$  are the initial values at  $\theta(\tau_0) = 0$ .

The proof consists of three steps: (i) We show that, if  $\varepsilon$  is set formally to zero, then the solutions  $(\delta R^0)(\theta), (\delta \bar{\phi}^0)(\theta)$  (in the latter the “secular term” has been removed, see (200) below) of the variational equation around  $X_{L0} \equiv \eta_{00L}(\tau)\varepsilon^{1/8}$  approach a limit as  $\tau$  increases indefinitely; (ii) For finite  $\varepsilon$  we show that, if  $\tau > \varepsilon^{-\alpha}$ , for some  $\alpha < 3/8$ , the solutions  $(\delta R)(\theta), (\delta \bar{\phi})(\theta)$  differ by arbitrarily small amounts from their values at  $t = \pi/2$ , provided  $\varepsilon$  is appropriately small; (iii) We show that, for an interval of values of  $\tau, \tau_0 < \tau < \varepsilon^{-\beta}$ , with  $3/8 > \beta > \alpha$ , the difference between the solutions  $\delta R^0(\theta), \delta \bar{\phi}^0(\theta)$  (corresponding to  $\varepsilon = 0$ ) and  $\delta R(\theta), \delta \bar{\phi}(\theta)$  (corresponding to a finite  $\varepsilon$ ) may be made as small as we wish, by letting  $\varepsilon$  be appropriately small.

In proving (i), we append for clarity to all variables related to the variational equation a superscript “0”, recalling that  $\varepsilon$  is set equal to zero. We assume we have “removed” by successive transformations the terms of  $O((k^0)^3)$  and  $O(dk^0/d\theta)$  of the dependent variables,  $(k^0(\theta) \equiv 1/\eta_{00}(\theta)^{3/2})$  and consider first the variational equation around the solutions  $R_{4L,0}, \phi_{4L,0}$  (Eqs. (D.4a), (D.4b)):

$$\begin{aligned} \frac{d(\delta R_4^0)}{d\theta} &= k^0(\theta)^4 R_{L,0}(\theta)^4 \bar{P}_1(z_0)(\delta R^0)(\theta) \\ &+ k^0(\theta)^4 R_{L,0}(\theta)^5 \bar{P}_2(z_0)(\delta \phi^0)(\theta) + \dots \end{aligned} \quad (190a)$$

$$\begin{aligned} \frac{d(\delta \phi_4^0)}{d\theta} &= -\frac{7}{24} \delta(k^0(\theta)^2 R_{L,0}(\theta)^2) \\ &+ k^0(\theta)^4 R_{L,0}(\theta)^3 \bar{Q}_1(z_0)(\delta R^0)(\theta) \\ &+ k^0(\theta)^4 R_{L,0}(\theta)^4 \bar{Q}_2(z_0)(\delta \phi^0)(\theta) + \dots \end{aligned} \quad (190b)$$

where  $\bar{P}_{1,2}, \bar{Q}_{1,2}$  are trigonometric polynomials of  $z_0 = \theta + \phi_{L,0}$  and the dots stay for terms which fall off more rapidly with  $\theta$ . It is relevant that the polynomials  $\bar{P}_1(z_0), \bar{P}_2(z_0), \bar{Q}_2(z_0)$  have zero mean whereas  $\bar{Q}_1(z_0)$  contains a constant (“secular”) term. From the form of the transformations leading from  $R_{L,0}, \phi_{L,0}$  to  $R_{4L,0}, \phi_{4L,0}$  (cf. Eqs. (121a), (121b), (123a), (123b), (125a), (125b)) we see that:

$$\begin{aligned} \delta R^0(\theta) &= \delta R_4^0(\theta)(1 + O(k^0(\theta))) \\ &+ \delta \phi_4^0(\theta)O(k^0(\theta)) \end{aligned} \quad (191a)$$

$$\begin{aligned} \delta \phi^0(\theta) &= \delta R_4^0(\theta)O(k^0(\theta)) \\ &+ \delta \phi_4^0(\theta)(1 + O(k^0(\theta))). \end{aligned} \quad (191b)$$

In (191a), (191b) the terms denoted by  $O(k^0)$  may be read off (123a), (123b): they contain trigonometric polynomials with zero mean. It follows that the variation of the secular term in (190b) is

$$\begin{aligned} \delta(k^0(\theta)^2 R_{L,0}(\theta)^2) &= 2k^0(\theta)^2 R_{L,0}(\theta) \delta R_4^0(\theta) \\ &+ (O(k^0(\theta)^3) \delta \phi_4^0(\theta)). \end{aligned} \quad (192)$$

These terms also give the leading order (the terms with the slowest falloff in  $\theta$ ) of the coefficients of  $\delta R_4^0(\theta), \delta \phi_4^0(\theta)$  in (190b).

We show next that the solution  $\delta R_4^0(\theta)$  of Eq. (190a) is actually bounded as  $\theta \rightarrow \infty$  and the solution  $\delta \phi_4^0$  of Eq. (190b) obeys  $\delta \phi_4^0 = O(\theta^{1/4})$  in the same limit. We consider to this end the Lyapunov expression:

$$L = r(\delta R_4^0(\theta))^2 + \frac{s(\delta \phi_4^0(\theta))^2}{\theta^{1/2}} \quad (193)$$

where  $r, s > 0$  are two parameters which may be chosen freely. Denoting by  $a_R(\theta), a_P(\theta)$  and  $b_R(\theta), b_P(\theta)$  the coefficients of  $\delta R_4^0, \delta \phi_4^0$  in Eqs. (190a), (190b) in turn, we may write, after taking (192) into account:

$$\begin{aligned} \frac{1}{2} \frac{dL}{d\theta} &= r a_R(\theta)(\delta R_4^0)^2 \\ &+ \left( r a_P(\theta) + s \frac{b_R(\theta)}{\theta^{1/2}} \right) (\delta R_4^0)(\delta \phi_4^0) \\ &+ s \frac{b_P(\theta)}{\theta^{1/2}} (\delta \phi_4^0)^2 - \frac{1}{4} s \frac{(\delta \phi_4^0)^2}{\theta^{3/2}}. \end{aligned} \quad (194)$$

Using now the inequalities:

$$|\delta R_4^0| \leq \sqrt{\frac{L}{r}} \quad |\delta \phi_4^0| \leq \theta^{1/4} \sqrt{\frac{L}{s}} \quad (195)$$

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and the fact that the last term in Eq. (194) is negative, we deduce:

$$\frac{dL}{d\theta} \leq 2L \left( |a_R(\theta)| + \sqrt{\frac{r}{s}} \theta^{1/4} |a_P(\theta)| + \sqrt{\frac{s}{r}} \frac{|b_R(\theta)|}{\theta^{1/4}} + b_P(\theta) \right). \quad (196)$$

For large  $\theta$  the dominant term is the one containing  $|b_R(\theta)|/\theta^{1/4}$  as one sees by reverting to Eqs. (190a), (190b) and (192): it falls off like  $1/\theta$ . Inequality (196) can be integrated to yield

$$L < \text{const} \times \theta^{2C} \sqrt{s/r}, \quad C = \sup |b_R(\theta)| \theta^{3/4}.$$

We may choose  $r$  and  $s$  so that the exponent of  $\theta$  be  $2d$ , with  $0 < d < 1/8$ . It follows then from Eq. (195) that:

$$|\delta\phi_4^0(\theta)| < \text{const} \times \theta^{d+1/4}. \quad (197)$$

We now return to Eq. (190a) and integrate it from an initial value  $\theta_i$  to some  $\theta$ : since  $\delta\phi_4^0$  obeys (197), the integral over  $\delta\phi_4^0$  is convergent and we deduce from Gronwall's inequality that:

$$\delta R_4^0(\theta) < \text{const} \times \exp \left( \int_{\theta_i}^{\theta} |a_R(\theta)| d\theta \right) < \text{const}. \quad (198)$$

Thus  $\delta R_4^0(\theta)$  is bounded for all  $\theta$ . Further, integrating Eq. (190a) between two values  $\theta_m$  and  $\theta_n$  one sees that  $|\delta R_4^0(\theta_m) - \delta R_4^0(\theta_n)| \rightarrow 0$  as  $(\theta_m, \theta_n) \rightarrow \infty$  and thus  $\delta R_4^0(\theta)$  approaches a limit as  $\theta \rightarrow \infty$ . This limit is approached like  $1/\theta^{1/4-d}$  (the dominant term in (190a) is the one containing  $\delta\phi^0$ ). From (191a) and the fact that  $k^0(\theta) \approx 1/\theta^{3/8}$ , it follows that, if the exponent  $d$  in (197) is less than  $1/8$ ,  $\delta R^0(\theta)$  itself approaches a limit as  $\theta \rightarrow \infty$ . This approach is at least of  $O(1/\theta^{1/8-d})$ .

We apply next Gronwall's inequality to Eq. (190b), after taking (192) into account:

$$\begin{aligned} |\delta\phi_4^0(\theta)| &< \sup \left| \int_0^{\theta} k^0(\theta)^2 R_{L,0}(\theta) \delta R(\theta) d\theta \right| \\ &\times \exp \left( \text{const} \times \int_0^{\theta} k^0(\theta)^3 d\theta \right) \\ &< \text{const} \times \theta^{1/4} \end{aligned} \quad (199)$$

where we have used the fact that the integral in the exponent is convergent, whereas the factor in front is bounded by  $\text{const} \times \theta^{1/4}$  in view of Eq. (198). As

a consequence of (199) we may assume from now on that the small exponent  $d = 0$ . We define further:

$$\begin{aligned} \delta\bar{\phi}_4^0(\theta) &= \delta\phi_4^0(\theta) \\ &+ \frac{7}{12} \int_0^{\theta} k^0(\theta)^2 R_{L,0}(\theta) (\delta R^0)(\theta) d\theta. \end{aligned} \quad (200)$$

Taking into account Eqs. (191a), (191b), it is true that:

$$\begin{aligned} \frac{d\delta\bar{\phi}_4^0(\theta)}{d\theta} &= c_R(\theta)(\delta R_4^0)(\theta) + c_P(\theta)(\delta\phi_4^0)(\theta) \\ c_R(\theta), c_P(\theta) &= O(k^0(\theta)^4). \end{aligned} \quad (201)$$

We integrate Eq. (201) from an initial value  $\theta_i$  to  $\theta$  and use the bounds of Eqs. (198), (199) to deduce that  $\delta\bar{\phi}_4^0(\theta)$  is itself bounded and has a limit as  $\theta \rightarrow \infty$ . We can now revert to an "original" variation of the "rest phase"  $\delta\bar{\phi}^0$  defined by:

$$\delta\bar{\phi}^0 = \delta\phi^0 + \frac{7}{12} \int_0^{\theta} k^0(\theta)^2 R_{L,0}(\theta) \delta R^0(\theta) d\theta. \quad (202)$$

Since  $\delta\phi^0$  is related to  $\delta\phi_4^0$  by (191b) and  $\delta\bar{\phi}_4^0$  defined in (200) was shown to be bounded, it follows that  $\delta\bar{\phi}^0$  is itself bounded and has a limit as  $\theta \rightarrow \infty$ , which is approached like  $1/\theta^{1/8}$  (or  $1/\tau^{1/6}$ ). We denote the limits of  $\delta R^0(\theta)$ ,  $\delta\bar{\phi}^0(\theta)$  by  $\delta R_f^0$ ,  $\delta\bar{\phi}_f^0$ .

We now turn to point (ii) of the proof and show for finite  $\varepsilon$  that, if  $\tau > \varepsilon^{-\alpha}$  with  $0 < \alpha < 3/8$ , the difference between  $\delta R(\theta(\tau); \varepsilon) \equiv \delta R(\theta)$  and its value at  $\theta(t = \pi/2)$  becomes vanishingly small as  $\varepsilon \rightarrow 0$ . The same is true for  $\delta\bar{\phi}(\theta; \varepsilon) \equiv \delta\bar{\phi}(\theta)$  of (202). This is done by repeating the argument above, but using the complete function  $k(\theta) \equiv k(\theta, \varepsilon)$  for finite  $\varepsilon$  and correspondingly a modified Lyapunov function  $\mathcal{L}(\theta)$ :

$$\begin{aligned} \mathcal{L}(\theta) &= r(\delta R_4(\theta))^2 + s \frac{\delta\phi_4(\theta)^2}{\Phi(\theta)^2}, \\ \Phi(\theta) &\equiv \int_0^{\theta} k(\theta)^2 d\theta. \end{aligned} \quad (203)$$

The function  $\Phi(\theta)$  increases like  $\theta^{1/4}$  for  $\tau < 1/\gamma$  and then stays approximately constant at values of  $O(1/\gamma^{1/3})$ . All arguments used above for  $\varepsilon = 0$  may be repeated with the conclusion that

$$\begin{aligned} |\delta R_4(\theta)| &< \text{const} \\ |\delta\phi_4| &< \text{const} \times \Phi(\theta) < \text{const} \times \theta^{1/4} \end{aligned} \quad (204)$$



and that the differences to the values at  $t = \pi/2$  obey:

$$\begin{aligned} & \left| \delta R_4(\tau = \varepsilon^{-\alpha}) - \delta R_4\left(t = \frac{\pi}{2}\right) \right|, \\ & \left| \delta \bar{\phi}_4(\tau = \varepsilon^{-\alpha}) - \delta \bar{\phi}_4\left(t = \frac{\pi}{2}\right) \right| \leq C\varepsilon^{\alpha/3}. \end{aligned} \quad (205)$$

Returning to the original  $\delta R(\theta), \delta \bar{\phi}$ , these bounds are turned into:

$$\begin{aligned} & \left| \delta R(\tau = \varepsilon^{-\alpha}) - \delta R\left(t = \frac{\pi}{2}\right) \right|, \\ & \left| \delta \bar{\phi}(\tau = \varepsilon^{-\alpha}) - \delta \bar{\phi}\left(t = \frac{\pi}{2}\right) \right| \leq C\varepsilon^{\alpha/6}. \end{aligned} \quad (206)$$

We turn now to point (iii) of the argument and compare directly to the values of  $\delta R(\theta, \varepsilon), \delta \bar{\phi}(\theta, \varepsilon)$  with those obtained for  $\varepsilon = 0$ . To this end, we revert again to Eqs. (190a), (190b) and (201) written appropriately for  $\varepsilon = 0$  and a finite small  $\varepsilon$  value. We subtract them and using notations like, e.g.  $\Delta(\delta R_4(\theta)) \equiv \delta R_4(\theta, \varepsilon) - \delta R_4^0(\theta)$  and similarly for  $\Delta(\delta \bar{\phi}_4), \Delta(\delta \phi_4)$  and also  $\Delta a_R(\theta) \equiv a_R(\theta, \varepsilon) - a_R(\theta, 0)$ , etc. (cf. (196)) we obtain equations of the form:

$$\begin{aligned} \frac{d\Delta(\delta R_4)}{d\theta} &= \Delta a_R(\theta) \delta R_4(\theta) + \Delta a_P(\theta) \delta \phi_4(\theta) \\ &+ a_R^0(\theta) \Delta \delta R_4(\theta) + a_P^0(\theta) \Delta \delta \phi_4(\theta) \end{aligned} \quad (207a)$$

$$\begin{aligned} \frac{d\Delta(\delta \phi_4)}{d\theta} &= \Delta b_R(\theta) \delta R_4(\theta) + \Delta b_P(\theta) \delta \phi_4(\theta) \\ &+ b_R^0(\theta) \Delta \delta R_4(\theta) + b_P^0(\theta) \Delta \delta \phi_4(\theta). \end{aligned} \quad (207b)$$

These equations are an inhomogeneous version of Eqs. (190a), (190b). The solutions of the homogeneous part have been shown to obey the bounds of Eqs. (197), (198). By the method of “variation of the constants”, we may write qualitatively a general solution of Eqs. (207a), (207b): since the initial conditions are  $\Delta \delta R_4(0) = \Delta \delta \phi_4(0) = 0$  (we are interested in those solutions that obey, e.g.  $\partial R / \partial R_0(\theta = 0) = 1$ , independently of  $\varepsilon$ ), we obtain, e.g.

$$\begin{aligned} \Delta \delta R(\theta) &= \int_0^\theta (f_R(\theta')(\delta \phi^{(2)})(\theta')) \\ &- f_\phi(\theta')(\delta R^{(2)})(\theta')) d\theta' (\delta R^{(1)})(\theta) \end{aligned}$$

$$\begin{aligned} & - \int_0^\theta (f_R(\theta')(\delta \phi^{(1)})(\theta')) \\ & - f_\phi(\theta')(\delta R^{(1)})(\theta')) d\theta' (\delta R^{(2)})(\theta), \end{aligned} \quad (208)$$

where  $f_R(\theta), f_\phi(\theta)$  represent the inhomogeneous terms in Eqs. (207a), (207b) and  $\delta R^{(i)}, \delta \phi^{(i)}, i = 1, 2$  are two independent solutions of the homogeneous equation with a wronskian equal to unity. As an example of an estimate of the differences  $\Delta a_R, \Delta b_R$ , etc. we consider:

$$\begin{aligned} \Delta b_R(\theta) &= O\left(\frac{1}{\tau} - \frac{\exp(-\gamma\tau)}{\tau(1 + \varepsilon^{3/4}\tau^2)}\right) \\ &= O(\varepsilon^{3/4}\tau), \quad \tau < \frac{1}{\gamma}. \end{aligned} \quad (209)$$

These differences are to be evaluated at fixed  $\theta$ , i.e. the values of  $\tau$  appearing in the two terms in (209) are *a priori* different; however, as shown in Appendix D (cf. Eq. (D.12)) it is enough for coarse estimates to use in both terms the value of  $\tau$  corresponding to  $\varepsilon = 0$ , if  $\tau < \varepsilon^{-\delta}$  with  $\delta < 3/8$ . Using Eqs. (197), (198) one verifies that:

$$f_R(\theta) = O(\varepsilon^{3/4}\tau^{1/3}), \quad f_\phi(\theta) = O(\varepsilon^{3/4}\tau). \quad (210)$$

With this we estimate from Eq. (208) and its analogue for  $\Delta \delta \phi_4$ :

$$\begin{aligned} \Delta \delta R_4(\theta) &= O(\varepsilon^{3/4}\theta^{7/4}), \\ \Delta \delta \phi_4(\theta) &= O(\varepsilon^{3/4}\theta^2). \end{aligned} \quad (211)$$

Taking into account Eqs. (191a), (191b), we verify that these estimates hold unchanged even for  $\Delta \delta R(\theta), \Delta \delta \phi(\theta)$ . Using Eq. (202), we obtain:

$$\Delta \delta \bar{\phi} = O(\varepsilon^{3/4}\theta^{7/3}).$$

If  $\tau < \varepsilon^{-\beta}$ , ( $\theta < \varepsilon^{-4\beta/3}$ ),  $\Delta \delta R, \Delta \delta \bar{\phi}$  tend to zero with  $\varepsilon$  provided  $3/4 - 28\beta/9 < 0$ ; this is fulfilled if, e.g.  $\beta = 1/5$ . For any choice of  $\alpha < 1/5$ , the difference between  $\delta R, \delta \bar{\phi}(\tau)$  and their values at  $t = \pi/2$  vanishes as  $\varepsilon \rightarrow 0$  (cf. Eqs. (205), (206)). According to (i)  $\delta R^0(\theta), \delta \bar{\phi}^0(\theta)$  approach their asymptotic values like  $1/\theta^{1/8}$  and thus the differences to these latter are  $O(\varepsilon^{\alpha/6})$  at  $\tau = \varepsilon^{-\alpha}$ . We can thus conclude the limiting values  $(\delta R(t = \pi/2), \delta R^0), (\delta \bar{\phi}(t = \pi/2), \delta \bar{\phi}^0)$  also approach each other as  $\varepsilon \rightarrow 0$ . This ends the proof of Lemma 6.4.

### 6.3. The mapping $\mathbb{DP}_R$

The first derivative  $\mathbb{DP}_R$  of the Poincaré map is obtained from the values at  $\tau = \pi/(2\varepsilon^{3/8})$  of

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two special solutions  $\delta R(\theta), \delta \phi(\theta)$  of the variational equation, with initial conditions at  $\tau = \tau_0$ :  $(\delta R = 1, \delta \phi = 0), (\delta R = 0, \delta \phi = 1)$  in turn. According to Lemma 6.4, the values  $\partial R / \partial R_0(\pi / (2\varepsilon^{3/8})), \partial R / \partial \phi_0(\pi / (2\varepsilon^{3/8}))$  approach as  $\varepsilon \rightarrow 0$  the asymptotic values  $(\partial R / \partial R_0)^0, (\partial R / \partial \phi_0)^0$  of the solutions of an equation in which  $\varepsilon$  was set formally equal to zero (we recall  $R_0 \equiv R(\tau_0), \phi_0 \equiv \phi(\tau_0)$ , with  $R, \phi$  of Eqs. (182a), (182b)). These values turn out to be:

$$\begin{aligned} A_R^0 &\equiv \left( \frac{\partial R}{\partial R_0} \right)^0 \cong 0.96 \\ A_P^0 &\equiv \left( \frac{\partial R}{\partial \phi_0} \right)^0 \cong 0.08. \end{aligned} \quad (212)$$

The same is true for the asymptotic values of the derivatives of the “rest phase”  $\bar{\phi}^0$

$$\begin{aligned} B_R^0 &\equiv \left( \frac{\partial \bar{\phi}}{\partial R_0} \right)^0 \cong 0.25 \\ B_P^0 &\equiv \left( \frac{\partial \bar{\phi}}{\partial \phi_0} \right)^0 \cong 1.08. \end{aligned} \quad (213)$$

These values depend on the point  $\tau_0$  which, for numerical convenience, is chosen sufficiently large so that the asymptotic form  $\eta(\tau) \approx \tau^{1/3}$  be valid.<sup>20</sup> It turns out that the convergence of the variations

$\delta R(\theta)$  to the limiting values is rapid, that of the  $\delta \bar{\phi}$  is, however, very slow.

On the other hand, the derivative  $\partial \phi / \partial R_0(\theta)$  diverges as  $\theta \rightarrow \infty$ . This shows the origin of the asymptotic circle map given by the Duffing equation: the small disk of radius  $\varepsilon^{\kappa\pi}$  is “stretched” in a  $\tau$ -time interval of the order  $1/\gamma$  in a rectangle in the  $(R, \phi)$  plane, highly elongated in the  $\phi$ -direction. In a  $\tau$ -interval of  $O(1/\gamma)$ , friction plays no role: this stretching is entirely “Hamiltonian”: the volume in phase space is conserved (the transformation (182a), (182b) is not canonical; for a unit Jacobian, one must multiply  $\delta \phi$  by  $R$ ; the area of the rectangle is multiplied by a factor  $R(\tau = 1/\gamma)/R(\tau_0) \approx 1$ ). Clearly, when  $\phi$  is “wrapped” back on the unit circle, it will possibly cover it — depending on the size of the initial disk — more than once: it is this mapping of the circle into itself which leads to the chaotic motion observed at smaller values of the damping.<sup>21</sup> The arguments of this paper show that it must be observed at increasing values of the damping as the forcing  $\Gamma$  increases indefinitely.

In Fig. 8 we show the image of the circle  $\Lambda = 1$  at  $t = -\pi/2$  in the  $(w, dw/d\theta)$  plane at  $t = \pi/2$  for  $\varepsilon^{3/8} = 0.003, \kappa = 0.04$ ; one sees the extreme angular stretching (in the  $\phi$ -direction) caused by the diverging derivative  $\partial \phi / \partial R_0$ . The crosses show

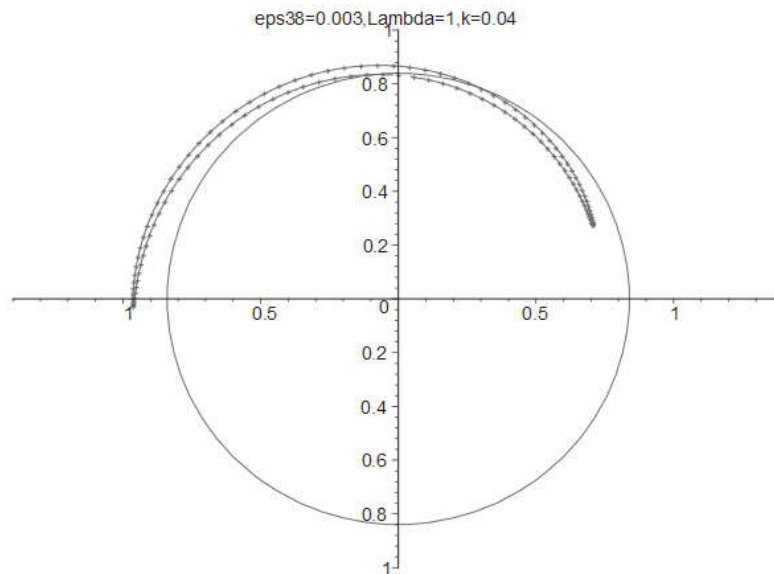


Fig. 8. The image of the circle  $\Lambda = 1$  at  $t = -\pi/2$  in the  $(w, dw/d\theta)$  plane at  $t = \pi/2$ .

<sup>20</sup> $\tau_0 = 10$  is a possible choice; clearly, its value drops out in the final results.

<sup>21</sup>As is shown in Sec. 7, chaotic motions appear long before the circle is completely covered.



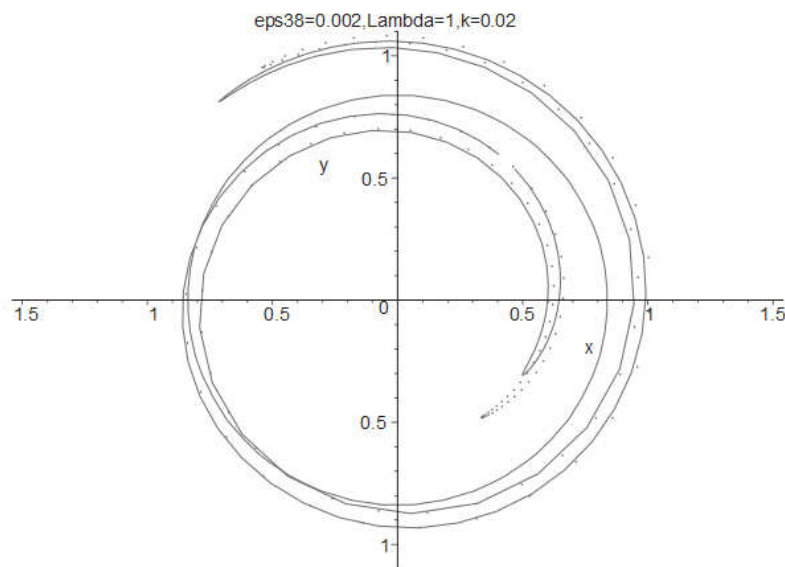


Fig. 9. The image of the circle  $\Lambda = 1$  at  $t = -\pi/2$  in the  $(w, dw/d\theta)$  plane at  $t = \pi/2$  at smaller damping.

the approximation offered by the variational equation (for  $\partial R, \phi/\partial R_0, \phi_0$ ). Figure 9 shows a situation at smaller damping ( $\varepsilon^{3/8} = 0.002, \kappa = 0.02$ ) where the stretching exceeds  $2\pi$ . At smaller  $\varepsilon$  (larger forcing) the arms of the spiral approach each other so that asymptotically the disk  $\Lambda = 1$  is mapped into a very thin ring at  $t = \pi/2$ . The points show again the approximation of the Poincaré plot by the first derivative; feeling supported by this numerical evidence,<sup>22</sup> we do not discuss in this paper at all the corrections due to higher terms of the Taylor expansion.

We evaluate next more carefully the derivatives  $\partial\phi/\partial R_0(\theta(\pi/2)), (\partial\phi/\partial\phi_0)(\theta(\pi/2))$  for finite, small values of  $\varepsilon$ .

$$\frac{\partial\phi}{\partial R_0} = \frac{\partial\bar{\phi}}{\partial R_0} - \frac{7}{12} \int_0^{\theta(\pi/2)} k(\theta)^2 R_{L,\varepsilon}(\theta) \frac{\partial R}{\partial R_0}(\theta) d\theta. \quad (214)$$

Both the first term and  $R_{L,\varepsilon}(\theta), (\delta R)(\theta)$  under the integral sign are bounded<sup>23</sup> at  $\theta = \theta(\pi/2)$ . The leading behavior of the phase is determined by the integral: according to Lemmas 6.1 and 6.4, the values of  $R_{L,\varepsilon}(\theta(\pi/2)), \partial R/\partial R_0(\theta(\pi/2))$  have limits as  $\varepsilon \rightarrow 0$ , denoted by  $R_{L,f}(0), A_R^0$  (vgl. Lemma 6.1 and (212)). Thus we may write for the dominant

contribution in (214):

$$\begin{aligned} \left| \left( \frac{\partial\phi}{\partial R_0} \right)_d \right| &= \left| -\frac{7}{12} R_{L,f}(0) A_R^0 \int_0^{\theta(\pi/2)} k(\theta)^2 d\theta \right| \\ &\leq \text{const} \times \int_0^{\pi/(2\varepsilon^{3/8})} \frac{\exp(-2\gamma\tau)}{\tau^{2/3}} d\tau \\ &= \frac{\text{const}}{\gamma^{1/3}} \end{aligned} \quad (215)$$

which shows explicitly the divergence as  $\varepsilon \rightarrow 0$  in the factor  $1/\gamma^{1/3}$ . It turns out that the factor multiplying  $1/\gamma^{1/3}$  has itself a slow logarithmic dependence on  $\varepsilon$  and approaches a finite value as  $\varepsilon \rightarrow 0$  ( $\approx 1.80$ ). Figure 10 shows the  $\varepsilon$ -dependence of the quantity  $C_0(\varepsilon)$  defined by comparison to (215) through

$$\left( \frac{\partial\phi}{\partial R_0} \right)_d = -\frac{C_0(\varepsilon) A_R^0}{\gamma^{1/3}}. \quad (216)$$

The remaining  $\kappa$ -dependence of  $C_0(\varepsilon)$  is negligible in our range ( $\varepsilon^{3/8}/\gamma^{1/3} \approx 1$ ). The order of magnitude of  $\partial\phi/\partial R_0$  is correctly reproduced if the integrals are simply cutoff at  $\tau = 1/\gamma$  and the exponential term in  $k(\theta)$  is ignored. For finite  $\varepsilon$  the expression in (215) should be multiplied by  $(1 + O(\varepsilon^s))$  for some small  $s$ , which controls the approach of  $\partial R/\partial R_0(\theta(\pi/2))$  to  $A_R^0$  as  $\varepsilon \rightarrow 0$  (see

<sup>22</sup>The agreement is better than warranted by the estimates of the paper.

<sup>23</sup> $\theta(\pi/2)$  means the  $\theta$ -value corresponding to  $t = \pi/2$ ; it is  $O(1/\varepsilon^{1/2})$ .

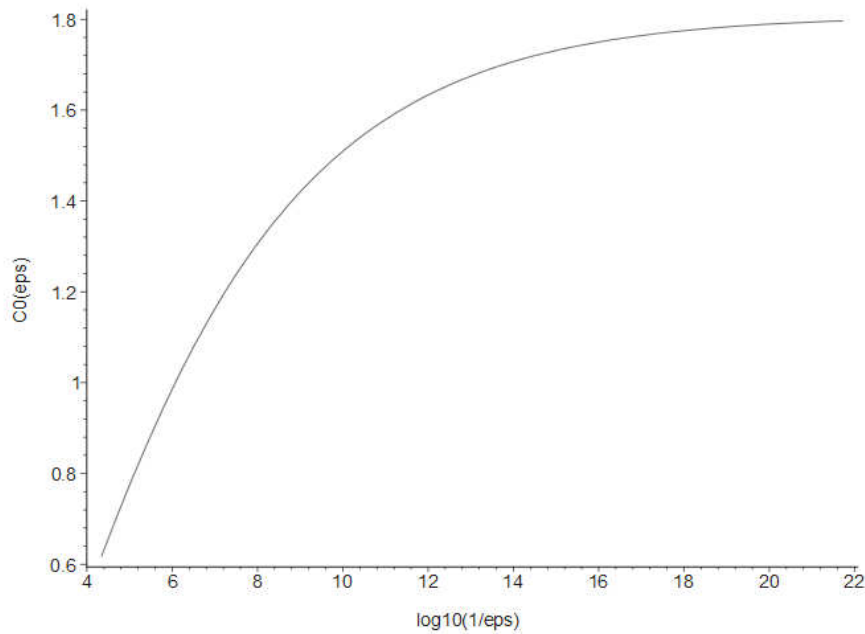


Fig. 10. The  $\varepsilon$ -dependence of the quantity  $C_0(\varepsilon)$  of Eq. (216).

previous section). We compute next (upper bounds to) the corrections to the dominant term and evaluate, for finite  $\varepsilon$ :

$$\mathcal{I} \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 R_{L,\varepsilon}(\theta) \frac{\partial R}{\partial R_0}(\theta) - R_{L,f}(\varepsilon) \frac{\partial R}{\partial R_0} \left( \theta \left( \frac{\pi}{2} \right) \right) d\theta. \quad (217)$$

In Appendix E we show that this integral is bounded and even has a limit, which we denote by  $D_R^0$ , as  $\varepsilon \rightarrow 0$ . This is in principle a finite quantity which must be added to the contribution of the “rest phase” (202). However, since  $R_{L,\varepsilon}(\theta)$ ,  $\partial R / \partial R_0(\theta)$  are, apart from small oscillations, remarkably constant, the contribution of (217) is  $\approx 0$ . The same arguments may be repeated for the partial derivative  $\partial \phi / \partial \phi_0$ : the leading term is  $O(1/\gamma^{1/3})$  in analogy to (215):

$$\begin{aligned} \frac{\partial \phi}{\partial \phi_0} \left( \theta \left( \frac{\pi}{2} \right) \right) &= -\frac{7}{12} \frac{\sqrt{3}}{2^{1/3} \gamma^{1/3}} \Gamma \left( \frac{1}{3} \right) R_{L,f}(0) A_P^0 \\ &\equiv -\frac{C_0(\varepsilon=0) A_P^0}{\gamma^{1/3}} \end{aligned} \quad (218)$$

with  $A_P^0$  of Eq. (212). There exists in principle an additional bounded term, which tends to a limit  $D_P^0$  as  $\varepsilon \rightarrow 0$ . It is the sum of  $\partial \bar{\phi} / \partial \phi_0$ , which approaches (slowly) the value  $B_P^0$ , Eq. (213) and of the analogon of (217), which is  $\approx 0$ . To conclude,

the map  $(R_0, \phi_0) \Rightarrow (R(\theta(\pi/2)), \phi(\theta(\pi/2)))$  is given by  $(R_{0L}, \phi_{0L})$  are the values of  $(R_0, \phi_0)$  assumed by  $X_L$  at  $\tau_0$ :

$$\begin{aligned} R_0 &\Rightarrow R_{L,f}(\varepsilon) + A_R(\varepsilon)(R_0 - R_{0L}) \\ &\quad + A_P(\varepsilon)(\phi_0 - \phi_{0L}) + \text{higher orders} \end{aligned} \quad (219a)$$

$$\begin{aligned} \phi_0 &\Rightarrow \phi_L \left( \frac{\pi}{2}, \varepsilon \right) \\ &\quad + \left( -\frac{C_0(\varepsilon) A_R(\varepsilon)}{\gamma^{1/3}} + C_R(\varepsilon) \right) (R_0 - R_{0L}) \\ &\quad + \left( -\frac{C_0(\varepsilon) A_P(\varepsilon)}{\gamma^{1/3}} + C_P(\varepsilon) \right) (\phi_0 - \phi_{0L}) \\ &\quad + \text{h.o.} \end{aligned} \quad (219b)$$

where the higher orders are  $O(\varepsilon^{2\kappa\pi})$  multiplied by the order of magnitude of the second derivative. The terms  $A_R(\varepsilon)$ ,  $A_P(\varepsilon)$  approach the values  $A_P^0$ ,  $A_R^0$  of Eq. (212). The terms  $C_R(\varepsilon)$ ,  $C_P(\varepsilon)$  are the sums of the corrections to the leading  $1/\gamma^{1/3}$  term given by (217) (and its analogon for  $\partial R / \partial \phi_0$ ) and the contributions (213) of the “rest phase”  $\bar{\phi}$  in (214). As  $\varepsilon \rightarrow 0$  these terms approach (slowly) the limiting values (213) given by the  $\varepsilon = 0$  equation:

$$\begin{aligned} C_R(\varepsilon) &\Rightarrow B_R^0 + D_R^0 \approx B_R^0 \\ C_P(\varepsilon) &\Rightarrow B_P^0 + D_P^0 \approx B_P^0. \end{aligned} \quad (220)$$



Equations (219a) and (219b) contain the matrix elements of  $\mathbb{D}\mathbb{P}_f$ , Eq. (187). The complete Jacobian  $\mathbb{D}\mathbb{P}_R$  evaluated at  $(\Delta\eta(0), d\Delta\eta/d\tau(0))$  is computed by performing the matrix multiplication in (187). The elements of  $\mathbb{D}\mathbb{P}_i$  are in the limit  $\varepsilon \rightarrow 0$  given by the solutions of Eq. (188) with  $\gamma = 0$  and  $\eta_L$  replaced by  $\eta_{L00}$ . For reference, for the choice  $\tau_0 = 10$  they are:

$$\begin{aligned} \frac{\partial u_{0R}(\tau_0)}{\partial u_{0L}} &= 0.723 & \frac{\partial u_{0R}(\tau_0)}{\partial u'_{0L}} &= 1.111 \\ \frac{\partial u'_{0R}(\tau_0)}{\partial u_{0L}} &= 0.550 & \frac{\partial u'_{0R}(\tau_0)}{\partial u'_{0L}} &= 2.230. \end{aligned} \quad (221)$$

The mapping  $\mathbb{D}\mathbb{T}$  for  $\tau = \tau_0$  is obtained from (181). The dominant terms are:

$$\begin{aligned} \mathcal{J}_1 &\equiv \frac{\partial R}{\partial u_{0L}} \left( \theta = \frac{\pi}{2} \right) \cong -0.43, \\ \mathcal{J}_2 &\equiv \frac{\partial R}{\partial u'_{0L}} \left( \theta = \frac{\pi}{2} \right) \cong -0.247. \end{aligned} \quad (222)$$

With this, the right-hand side Poincaré map reads (cf. Eq. (183)):

$$\begin{aligned} &\left( u_L(0), \frac{du_L}{d\tau}(0) \right) \\ &\Rightarrow \left( R_{L,f}(\varepsilon) + \mathcal{J}_1 u_L(0) + \mathcal{J}_2 \frac{du_L}{d\tau}(0) + O(\varepsilon^{2\kappa\pi}), \right. \\ &\quad \theta \left( \frac{\pi}{2} \right) - \theta(\tau_0) + \phi_L \left( \frac{\pi}{2}, \varepsilon \right) \\ &\quad \left. - \frac{C_0}{\gamma^{1/3}} \left( \mathcal{J}_1 u_L(0) + \mathcal{J}_2 \frac{du_L}{d\tau}(0) \right) \right) \end{aligned}$$

$$+ E_0 u_L(0) + F_0 \frac{du_L}{d\tau}(0) + O \left( \frac{\varepsilon^{2\kappa\pi}}{\gamma^{1/3}} \right) \quad (223)$$

where  $\phi_L(\pi/2, \varepsilon)$  is defined through (175a), (175b); it contains a “secular”, divergent term  $\Phi_{L,R}(\theta(\pi/2))$  described in Lemma 6.2 (cf. Eq. (177)) and a “finite” term  $\bar{\phi}_L(\pi/2)$  which has a limit  $\bar{\phi}_{L,f}(0)$  as  $\varepsilon \rightarrow 0$ , as described in Lemma 6.3. Further,  $E_0, F_0$  approach as  $\varepsilon \rightarrow 0$  constants obtained from the values in Eqs. (220), (221):  $E_0 \approx 1.02$ ,  $F_0 \approx 2.11$ . The term  $O(\varepsilon^{2\kappa\pi}/\gamma^{1/3})$  is (only qualitatively) justified as follows: the second derivative of the Poincaré map is expected to diverge like  $\gamma^{-1/3}$  as  $\varepsilon \rightarrow 0$ , like the phase  $\phi_L$  and its first derivative  $\delta\phi$ , and is multiplied by terms of  $O(u^2) = O(\varepsilon^{2\kappa\pi})$ ; since — as may already be apparent — we expect bifurcations to occur when  $\varepsilon^{\kappa\pi}/\gamma^{1/3}$  is  $O(1)$ , this correction is of the same order of magnitude as the terms preceding it. However, for small  $\varepsilon$ , the bifurcation pattern is determined by the divergent terms (see Secs. 7 and 8).

## 7. The Complete Map $\mathbb{P}$ and Its Associated Circle Map

### 7.1. The map $\mathbb{P}$

We put now together Eqs. (166a), (166b) and (223) to obtain the image of a point

$$\mathbb{P} : \left( w, \frac{dw}{d\theta} \right) = (\Lambda \cos \Psi_0, -\Lambda \sin \Psi_0)$$

situated in a disk around  $(X_L(-\pi/2), dX_L/d\theta_L(-\pi/2))$  under the mapping  $\mathbb{P}$  of (80) (with the change  $t \Rightarrow \theta_{L,R}$  of the independent variable). We may write it as:

$$\begin{aligned} \Lambda \Rightarrow & R_{L,f}(\varepsilon) + \Lambda \varepsilon^{\kappa\pi} \left\{ \left( \mathcal{J}_1 V_c(0) + \mathcal{J}_2 \frac{dV_c}{d\tau}(0) \right) \cos(\Psi_0 + \Phi_L(\Lambda) + \Omega) \right. \\ & \left. - \left( \mathcal{J}_1 V_s(0) + \mathcal{J}_2 \frac{dV_s}{d\tau}(0) \right) \sin(\Psi_0 + \Phi_L(\Lambda) + \Omega) \right\} + O(\varepsilon^{\kappa\pi+\delta}) \end{aligned} \quad (224a)$$

$$\begin{aligned} \Psi_0 \Rightarrow & \pi + \theta_R \left( \frac{\pi}{2} \right) - \theta_R(\tau_0) + \phi_L \left( \frac{\pi}{2}, \varepsilon \right) - C_0 \Lambda \frac{\varepsilon^{\kappa\pi}}{\gamma^{1/3}} \left\{ \left( \mathcal{J}_1 V_c(0) + \mathcal{J}_2 \frac{dV_c}{d\tau}(0) \right) \cos(\Psi_0 + \Phi_L(\Lambda) + \Omega) \right. \\ & \left. - \left( \mathcal{J}_1 V_s(0) + \mathcal{J}_2 \frac{dV_s}{d\tau}(0) \right) \sin(\Psi_0 + \Phi_L(\Lambda) + \Omega) \right\} + \Lambda \varepsilon^{\kappa\pi} \left\{ \left( E_0 V_c(0) + F_0 \frac{dV_c}{d\tau}(0) \right) \right. \\ & \left. \times \cos(\Psi_0 + \Phi_L(\Lambda) + \Omega) - \left( E_0 V_s(0) + F_0 \frac{dV_s}{d\tau}(0) \right) \sin(\Psi_0 + \Phi_L(\Lambda) + \Omega) \right\} + O \left( \frac{\varepsilon^{2\kappa\pi}}{\gamma^{1/3}} \right) \end{aligned} \quad (224b)$$

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where we have used the notation in (223). In the region where  $\varepsilon^{\kappa\pi}/\gamma^{1/3}$  is  $O(1)$  the last two terms in (224b) are both of  $O(\varepsilon^{\kappa\pi})$  and will be treated together. These expressions are simplified by introducing  $\mathcal{M}$  and  $\xi$  through:

$$\begin{aligned}\mathcal{J}_1 V_c(0) + \mathcal{J}_2 \frac{dV_c}{d\tau}(0) &\equiv \mathcal{M} \cos \xi \\ \mathcal{J}_1 V_s(0) + \mathcal{J}_2 \frac{dV_s}{d\tau}(0) &\equiv \mathcal{M} \sin \xi.\end{aligned}\quad (225)$$

In Eqs. (224a), (224b) we wrote the  $\Lambda$ -dependence of  $\Phi_L$ , Eq. (131) explicitly. The term  $\pi$  in (224b) takes care of the minus sign present in the definition of the *half-period* map  $\mathbb{P}$  (cf. the definition Eq. (82)). We perform next the  $\varepsilon$ ,  $\gamma$ -dependent transformation of the angular variable in (224a), (224b):

$$\begin{aligned}\Psi_0 &= \chi + \pi + \theta_R\left(\frac{\pi}{2}\right) - \theta_R(\tau_0) + \phi_L\left(\frac{\pi}{2}, \varepsilon, \gamma\right) \\ &\equiv S\chi\end{aligned}\quad (226)$$

with which the transformed mapping  $\mathcal{P}$ :

$$\mathcal{P} \equiv S^{-1}\mathbb{P}S \quad (227)$$

reads:

$$\begin{aligned}\mathcal{P}: \quad \Lambda &\Rightarrow R_{L,f}(\varepsilon) + O(\varepsilon^{\kappa\pi}) \\ \chi &\Rightarrow \beta \frac{\Lambda}{R_{L,f}(\varepsilon)} \cos(\chi + \tilde{\Sigma}(\varepsilon, \gamma, \Lambda)) + O(\varepsilon^{\kappa\pi})\end{aligned}\quad (228)$$

with the following notations:

$$\beta \equiv C_0 \mathcal{M} R_{L,f}(\varepsilon) \frac{\varepsilon^{\kappa\pi}}{\gamma^{1/3}} \quad (229)$$

and

$$\begin{aligned}\tilde{\Sigma}(\varepsilon, \gamma, \Lambda) &\equiv \theta_R\left(\frac{\pi}{2}\right) - \theta_R(\tau_0) + \Omega + \phi_{L,R}\left(\frac{\pi}{2}, \varepsilon\right) \\ &\quad + \xi(\varepsilon, \gamma) + \Phi_L(\varepsilon, \gamma, \Lambda).\end{aligned}\quad (230)$$

The first terms in Eq. (230) give (almost) the total  $\theta$  — variation from  $-\pi/2$  to  $\pi/2$  (cf. Eq. (162)):

$$\begin{aligned}\Delta\theta &= \theta_R\left(\frac{\pi}{2}\right) - \theta_R(\tau_0) + \Omega \\ &= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_{-\pi/2}^{\pi/2} |\sin t|^{1/3} dt + \Theta_0(\varepsilon)\end{aligned}\quad (231)$$

where  $\Theta_0(\varepsilon)$  has a finite limit when  $\varepsilon \rightarrow 0$ . For small  $\varepsilon$  Eq. (228) shows that, under  $\mathcal{P}$ ,  $\Lambda$  is squeezed to values near  $R_{0L}(\varepsilon)$  (as is seen in Figs. 8 and 9) and therefore the term  $\Phi_L(\varepsilon, \gamma, \Lambda)$  gets very close to  $\Phi_L(\varepsilon, \gamma, R_{L,f}(\varepsilon))$ . The factor  $R_{L,f}(\varepsilon)$  has a limit

$R_{L,f}(0)$  as  $\varepsilon \rightarrow 0$ , independent of  $\gamma$  (cf. Lemma 6.1 and Appendix D). The same is true for the quantity  $\mathcal{M}$  (cf. (225)) according to Lemma 6.4. It is natural then to expect that the *one-dimensional* mapping of the unit circle into itself:

$$\Pi: \quad \chi \Rightarrow \beta \cos(\chi + \Sigma) \quad (232)$$

with

$$\begin{aligned}\Sigma &\equiv \Delta\theta + \phi_L\left(\frac{\pi}{2}, \varepsilon\right) + \xi(\varepsilon, \gamma) \\ &\quad + \Phi_L(R_{L,f}(\varepsilon), \varepsilon, \gamma)\end{aligned}\quad (233)$$

contains the essential features of the bifurcation structure of the mapping  $\mathcal{P}$ , Eq. (228) and thus of  $\mathbb{P}$ . In the following subsection we present some relevant features of this mapping, which is otherwise well studied [Zeng & Glass, 1989; Collet & Eckmann, 1983] (this is the standard reference on one-dimensional mappings; however, the map (232) falls a little outside the class of maps considered there). In the next section, we discuss its relation to the real Poincaré mapping of the Duffing equation  $\mathcal{P}$  of (228).

Clearly, the mapping  $\Pi$  may show a bifurcation structure in the region of parameter space where  $\beta = O(1)$ . This justifies some of the statements made before concerning the orders of magnitude coming into play (see comments following Eqs. (224b) and (223)). Referring to the discussion in Sec. 6.3, especially to that accompanying Figs. 8 and 9, it is easy to give a “physical” rationale for the parameter  $\beta$ : in a  $\tau$ -time interval of order  $1/\gamma$  the motion is (almost) Hamiltonian and the original disk (ellipse) in the  $(u(0), du/d\tau(0))$  plane with radius of  $O(\varepsilon^{\kappa\pi})$  is stretched into an increasingly thin filament of increasing angular aperture wrapping itself around a circle of radius  $R_{L,f}(\varepsilon)$ . The angular aperture increases at the rate  $\theta^{1/4}$  (cf. Eq. (199)), where:

$$\theta \approx \sqrt{3} \int_{-\pi/2}^{\pi/2} \tau'^{1/3} d\tau' \approx \frac{3\sqrt{3}}{4} \tau^{4/3}.$$

In terms of  $\theta$  the  $\tau$ -time  $1/\gamma$  is  $O(1/\gamma^{4/3})$  so that the original aperture of  $O(\varepsilon^{\kappa\pi})$  becomes in a  $\tau$ -time  $1/\gamma$  of  $O(\varepsilon^{\kappa\pi} \times 1/\gamma^{1/3})$  which is precisely the order of magnitude of  $\beta$  in (230). At  $\tau$ -times larger than  $1/\gamma$  the angular aperture does not increase any more considerably, but the area of the “wrapped” rectangle decreases simply due to the damping by a factor  $\approx \varepsilon^{\kappa\pi}$ .



## 7.2. The circle map $\Pi$

In this section we gather some properties of the map  $\Pi$  of Eq. (232). Clearly its features are periodic in  $\Sigma$ . From the definition of the latter in Eq. (233) the dominant term for small  $\varepsilon$  is  $\Delta\theta$  of (231) which behaves like  $1/\sqrt{\varepsilon}$ , i.e. like  $\Gamma^{1/3}$ . Thus, we expect the bifurcation pattern in the  $\Gamma - \Delta$  plane to have at high forcing an increasingly better periodicity in  $\Gamma^{1/3}$ . At fixed  $\varepsilon$ ,  $\beta$  decreases with increasing  $\gamma$ , i.e. with increasing  $\Delta$ . For a comparison with “normal” bifurcation plots, we draw bifurcation lines in a  $\Sigma - (-\beta)$  plane.

(i) For  $\beta < 1$  (high damping), the equation:

$$\beta \cos(\chi + \Sigma) = \chi \quad (234)$$

has, for all  $\Sigma$ , only one solution  $\chi_s$ . This solution is a *stable* fixed point of  $\Pi$  since  $|d\Pi/d\chi(\chi_s)| < 1$ . Even more,

**Lemma 7.1.** *The solution  $\chi_s$  of (234) is the only invariant set under  $\Pi$  if  $\beta < 1$ .*

Indeed,  $|d\Pi/d\chi| < 1$  for all  $\chi$ , so that the distance between any two  $\chi_1, \chi_2$  is contracted under  $\Pi$  if  $\beta < 1$ :

$$\begin{aligned} |\Pi(\chi_2) - \Pi(\chi_1)| &\leq \sup_{\chi \in (\chi_1, \chi_2)} |\Pi'(\chi)| |(\chi_2 - \chi_1)| \\ &\leq |(\chi_2 - \chi_1)|. \end{aligned} \quad (235)$$

Thus, the sequence of all iterates of any  $\chi$  under  $\Pi$  converges (to  $\chi_s$ ). (ii) If  $\beta < \pi$ ,  $\Pi$  maps the interval  $(-\pi, \pi)$  into itself, so that the theory of iterated mappings of intervals, as presented in [Collet & Eckmann, 1983] and [Guckenheimer & Holmes, 1983] may be directly taken over. If  $\Sigma = 0$ , the mappings  $\Pi(\beta, 0)$  are unimodal in the sense of Collet and Eckmann [1983]. The mappings  $\Pi(\beta, \Sigma = -\pi/2)$  apply  $[0, \pi]$ ,  $[-\pi, 0]$  into themselves and — if restricted to these intervals — make up a full family of unimodal maps [Collet & Eckmann, 1983, Section III.1, p. 174], for  $0 < \beta < \pi$ . Moreover, for all values of  $\Sigma$ , the functions  $f(\chi) = \Pi(\beta, \Sigma)(\chi)$  have a negative Schwarz derivative:

$$Sf \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 < 0. \quad (236)$$

For unimodal families with negative Schwarz derivative there exists a sequence of values  $\beta_1 < \beta_2 \dots$  which accumulates at a value  $\beta_c < \pi$  and for

which there exist *superstable* orbits of period  $2^p$ . For  $\beta = \beta_c$  there exists a (nonperiodic) attracting Cantor set for the action of  $\Pi(\beta, -\pi/2)$ . For larger values the motion may be “chaotic” (with *sensitive dependence on the initial conditions*). Thus, we expect that chaotic motion occurs in the Duffing equation before (i.e. at smaller  $\beta$ ) the Poincaré map covers the whole angular range  $0 < \phi < 2\pi$  (cf. Fig. 9). Even without the restriction to unimodal maps, i.e. for arbitrary choices of  $\Sigma$ , condition (236) places restrictions on the possible invariant sets of  $\Pi$  (see Lemma 7.2 below). (iii) At some fixed values of  $\Sigma$ , if we increase  $\beta$ , we reach a value beyond which Eq. (234) admits three (or more) solutions. The limiting values  $\beta_S$  are those for which the line  $\Pi = \chi$  is tangent to the graph of  $\Pi(\chi)$ , i.e.

$$\begin{aligned} \chi_S &= \beta_S \cos(\chi_S + \Sigma) \\ 1 &= -\beta_S \sin(\chi_S + \Sigma). \end{aligned} \quad (237)$$

It follows that:

$$\beta_S^2 = 1 + \chi_S^2 \geq 1 \quad (238)$$

and thus  $\beta_S = 1$  only if  $\chi_S = 0$ ; Eqs. (237) imply then  $\Sigma = -\pi/2 \pmod{2\pi}$ . The bifurcations occurring when  $\beta = \beta_S$  are — if  $\Sigma \neq -\pi/2$  — of *saddle-node* type: at neighboring larger values of  $\beta$ , two more solutions appear, corresponding to a stable and an unstable orbit of period 1 (under the action of  $\Pi$ ). If  $\Sigma = -\pi/2$ , the bifurcation at  $\beta = 1$  is of the *pitchfork* type: the unique solution existing at  $\beta < 1$  loses its stability and a pair of stable solutions of period 1 appear at  $\beta > 1$ .

(iv) From (237) one can obtain the exact form of the bifurcation line  $\beta = \beta_S(\Sigma)$ . Near  $\beta = 1$  it has a cusp: indeed, let in Eq. (237)  $\Sigma = -\pi/2 + \sigma$  so that (237) implies:

$$\tan(\chi_S + \sigma) = \chi_S. \quad (239)$$

For small  $\chi_S$  and  $\sigma$  this means, using (238):

$$\chi_S(\sigma) \approx (3\sigma)^{1/3} \quad \beta_S(\sigma) \approx 1 + (3\sigma)^{2/3} \quad (240)$$

which shows the cusp behavior. Figure 11 shows the saddle-node bifurcation lines in a  $\Sigma - (-\beta)$  plane (with an origin for  $\Sigma$  defined  $\pmod{2\pi}$ ).

(v) If at a fixed point  $\chi_F$  of  $\Pi(\chi)$ ,  $d\Pi/d\chi(\chi_F) = -1$ , the map  $\Pi$  has a *flip* bifurcation,<sup>24</sup> if certain

<sup>24</sup>See [Guckenheimer & Holmes, 1983, Chapter III, Theorem 3.5.1].

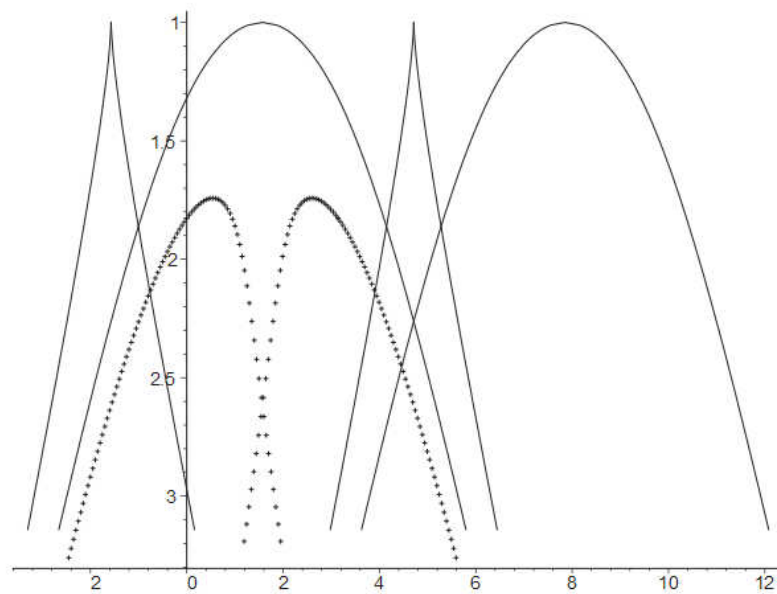


Fig. 11. The bifurcation lines in the  $\Sigma - (-\beta)$  plane.

transversality conditions are obeyed. In this case, these latter<sup>25</sup> are at fixed  $\Sigma$  simply:  $\chi_F^2 + 2 \neq 0$ ,  $\chi_F^2 + 2/3 \neq 0$ . The analogon of (237) is now:

$$\begin{aligned}\chi_F &= \beta_F \cos(\chi_F + \Sigma) \\ 1 &= \beta_F \sin(\chi_F + \Sigma)\end{aligned}\quad (241)$$

from which one sees that  $\beta_F(\Sigma) \geq 1$ , with equality only if  $\chi_F = 0$ , which occurs at  $\Sigma = \pi/2$ .

(vi) Equation (241) allows an exact determination of the flip bifurcation lines, see Fig. 11. Their maximum at the line  $\beta = 1$  is quadratic, which may be easily seen as follows: let  $\Sigma = \pi/2 + \sigma$  so that (241) implies:

$$\tan(\chi_F + \sigma) = -\chi_F \quad (242)$$

from which, for small  $\chi_F$ ,  $\sigma$ , one deduces:

$$\chi_F(\sigma) = -\frac{\sigma}{2} + O(\sigma^3). \quad (243)$$

Then (241) leads to (cf. (238)):

$$\beta_F(\sigma) \approx 1 + \frac{\sigma^2}{8}. \quad (244)$$

(vii) The flip bifurcation curves are much broader than the saddle-node ones (see Fig. 11). We give a simple estimate of the ratio of their widths (measured at the points where they intersect with the

smallest values of  $\beta$ ): at such points with coordinates  $(\Sigma^0, \beta^0)$ , the four equations (237), (241) (for the four unknowns  $\Sigma^0, \beta^0, \chi_S^0, \chi_F^0$ ) imply:

$$\chi_F^0 = \pm \chi_S^0 = \pm \sqrt{(\beta^0)^2 - 1}. \quad (245)$$

We look for solutions with  $\beta^0 < \pi$ , which means that  $|\chi_F^0|, |\chi_S^0| < \pi$ . The second of each pair of Eqs. (237), (241) exclude in turn the possibility  $\chi_F^0 = \chi_S^0$ . From each of the pairs (237), (241) one deduces that:

$$\cot(\chi_F^0 + \Sigma^0) = \chi_F^0 = -\chi_S^0 = \cot(\chi_S^0 + \Sigma^0) \quad (246)$$

so that:

$$\chi_F^0 = \chi_S^0 \pmod{\pi}.$$

With (245) this is possible only if  $\chi_S^0 = \pm\pi/2$ ,  $\chi_F^0 = \mp\pi/2$ . If  $\chi_S^0 = -\pi/2$ , (246) and (237) imply  $\Sigma^0 = -\arctan(\pi/2) \pmod{2\pi}$ . If  $\chi_S^0 = \pi/2$ , the solution accepted by (237) is  $\Sigma^0 = \pi + \arctan(\pi/2) \pmod{2\pi}$ . The ratio  $\rho$  of the length of two consecutive intervals between possible values of  $\Sigma^0$  is:

$$\rho = \frac{\pi - 2 \arctan\left(\frac{\pi}{2}\right)}{\pi + 2 \arctan\left(\frac{\pi}{2}\right)} \approx 0.22. \quad (247)$$

(viii) At a larger value of  $\beta$ ,  $\beta_2(\Sigma)$  the orbits of period two which appeared at the flip bifurcation

<sup>25</sup>Conditions (F1) and (F2) in [Guckenheimer & Holmes, 1983].



described above undergo another period-doubling bifurcation. Figure 11 shows the results of a numerical calculation of  $\beta_2(\Sigma)$ . As is well known in many instances, one can find values  $\beta_4(\Sigma)$ ,  $\beta_8(\Sigma)$  at which further bifurcations to periodic stable solutions of period 4, 8, ... etc. occur. For  $\Sigma = 0$ , one obtains  $\beta_2 = 1.8271$ ,  $\beta_4(0) = 1.9429$ ,  $\beta_8(0) = 1.9674 \dots$

(ix) For a complete description of the situation, one needs also an argument that, at least for values of  $\beta$  not too large, the mapping  $\Pi$  contains no other invariant sets apart from the fixed points (or orbits of period 2, etc.) described above. Because the family  $\Pi(\beta, \Sigma)$  does not fall entirely under the classes of one-dimensional mappings described in [Collet & Eckmann, 1983] and [Guckenheimer & Holmes, 1983], we give a statement which guarantees the (expected) absence of supplementary invariant sets for small enough  $\beta$ :

**Lemma 7.2.** *Let  $\beta_{2u}(\Sigma)$  be the second positive root (i.e. different from  $-\Sigma$ ) of the equation:*

$$\Pi(\beta, \Sigma; \chi = \beta) = \beta \cos(\beta + \Sigma) = -\Sigma \quad (248)$$

*if  $\Sigma < 0$  and the positive root of (248) if  $\Sigma > 0$  ( $-\pi < \Sigma < \pi$ ). Let  $\beta_{2d}(\Sigma)$  be the second positive root (i.e. different from  $\pi + \Sigma$ ) of:*

$$\begin{aligned} \Pi(\beta, \Sigma; \chi = -\beta) &= \beta \cos(-\beta + \Sigma) \\ &= -\pi - \Sigma \end{aligned} \quad (249)$$

*if  $\Sigma < 0$  and the first positive root of*

$$\begin{aligned} \Pi(\beta, \Sigma; \chi = -\beta) &= \beta \cos(-\beta + \Sigma) \\ &= \pi - \Sigma \end{aligned} \quad (250)$$

*if  $\Sigma > 0$ . Let:*

$$\beta_e(\Sigma) = \min[\pi, \beta_{2u}(\Sigma), \beta_{2d}(\Sigma)] \quad (251)$$

*Then, for  $0 < \beta < \beta_e(\Sigma)$  the invariant sets of  $\Pi(\beta, \Sigma)$  consist of at most three fixed points and two orbits of period two.*

The proof of this statement is relegated to Appendix F, because it is not very short; it uses the property of  $\Pi$  to have a negative Schwarz derivative (236) and owes a lot to the presentation in Ch. III of [Collet & Eckmann, 1983]. The conditions (248)–(250) describe *superstable* orbits of period two (i.e. orbits which pass through the maximum  $\beta$  or the minimum  $-\beta$  of  $\Pi(\beta, \Sigma; \chi)$ ). If e.g.  $\Sigma < 0$ , the maximum and minimum of  $\Pi$  occur at  $\chi_M = -\Sigma$ ,  $\chi_m = -\pi - \Sigma$  in turn,  $\Pi(\chi_M) = \beta$ ,

$\Pi(\chi_m) = -\beta$ : Eq. (248) states that the iteration of  $\chi_m$  under  $\Pi$  should repeat itself after two steps. For the root  $\beta_{1u} \equiv -\Sigma$  of (248) the fixed point of  $\Pi$  lies on the maximum of  $\Pi(\chi)$  (and on the minimum for  $\beta_{1d} \equiv \pi + \Sigma$  in (249).

This closes the qualitative discussion of the mapping  $\Pi$ .

## 8. The Bifurcations of Periodic Solutions at Large $\Gamma$ ! 0)

### 8.1. The bifurcations of $\Pi$ in the $\Gamma$ $\Delta$ plane

Instead of the parameters  $\Gamma$ ,  $\Delta$ , we can use  $\beta(\varepsilon, \gamma)$ ,  $\Sigma(\varepsilon, \gamma)$  of Eqs. (229), (233). If we believe that the map (232) reproduces the main features of the Poincaré map  $\mathbb{P}$ , then the tips of the bifurcation curves lie — alternately saddle-nodes and odd-periodic — simply periodic (flip) — along the line  $\beta = 1$ . Above certain critical values of  $\beta$  — we denote them by  $\hat{\beta}$  — there appear bifurcations to orbits with higher period. Assuming  $\varepsilon$  is so small that  $R_{0L}(\varepsilon)$ ,  $\mathcal{M}(\varepsilon)$  may be replaced with their limiting values at  $\varepsilon = 0$  (from Eqs. (167) and (222)  $\mathcal{M}_0 = 0.6814$ ), we obtain for the asymptotic form of the bifurcation lines of (232):

$$\begin{aligned} \Delta_c(\Gamma) &= \frac{1}{12\pi} \ln \Gamma - \frac{1}{3\pi} \ln \ln \Gamma \\ &\quad - \frac{1}{\pi} \ln \frac{\hat{\beta}}{\mathcal{M}_0 R_{L,f}(0) C_0} + O\left(\frac{\ln \ln \Gamma}{\ln \Gamma}\right) \end{aligned} \quad (252)$$

with  $R_{L,f}(0)$  of Lemma 6.1 and  $C_0$  of Eq. (216). Figure 12 gives an idea of the appearance of the lines  $\beta = \hat{\beta}$  in a  $\Delta/\ln \Gamma$  versus  $\ln \Gamma$  plot, with  $\hat{\beta} = 0.8$ , 1 (solid) and 2 and of the asymptotic approximation (dotted) of (252). The bifurcation structure of  $\Pi$ , Eq. (232) is periodic in  $\Sigma$ , i.e. the bifurcation pattern repeats itself along lines of fixed  $\kappa$  (Eq. (21)) in intervals  $\Delta\Gamma$  obeying:

$$\Sigma(\Gamma + \Delta\Gamma, \kappa \ln(\Gamma + \Delta\Gamma)) - \Sigma(\Gamma, \kappa \ln \Gamma) = 2\pi. \quad (253)$$

The dominant term in  $\Sigma$ , Eq. (233) is  $\Delta\theta$ , Eq. (231), which is proportional to  $1/\sqrt{\varepsilon}$ , i.e. to  $\Gamma^{1/3}$  (cf. Eq. (7)). If  $1/\sqrt{\varepsilon}$  is much larger than  $\gamma^{-1/3}$  — the magnitude of the second term  $\phi_L(\pi/2, \varepsilon)$  in (233) — the pattern repeats itself in equal intervals of  $\Gamma^{1/3}$ ,

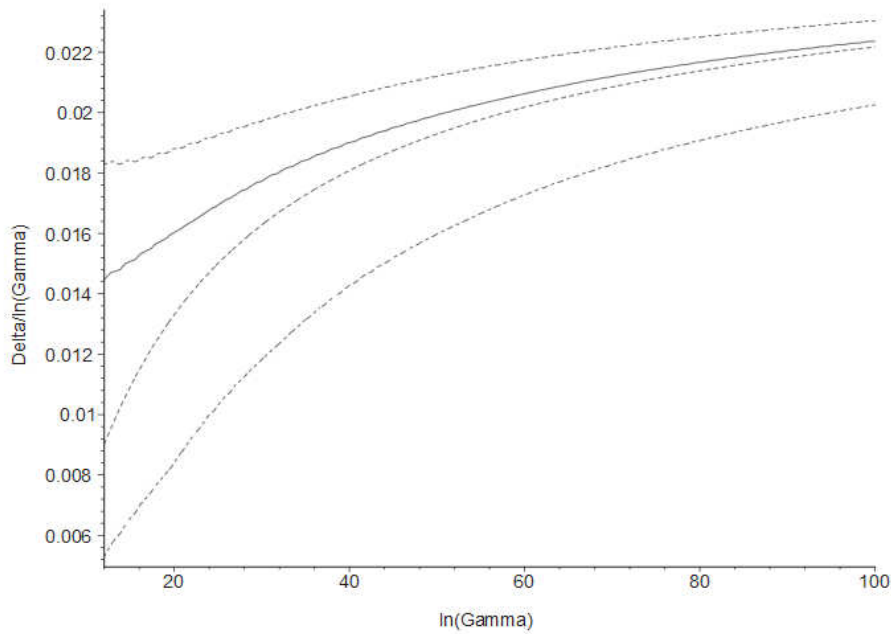


Fig. 12. The lines  $\beta = 0.8, 1$  (solid) and 2 and the approximation (252) to  $\beta = 1$  (dots).

independently of  $\kappa$ . The period is:

$$\Delta(\Gamma^{1/3}) = \frac{2\pi}{\sqrt{3}} \frac{1}{\int_{-\pi/2}^{\pi/2} |\sin t|^{1/3} dt} \cong 2.804. \quad (254)$$

Since the maxima of the saddle-node and flip bifurcation lines occur at  $\Sigma = -\pi/2, \pi/2 \pmod{2\pi}$ , they are asymptotically equidistant in  $\Gamma^{1/3}$  as shown

in Eq. (6). At smaller values of  $\kappa$  (or smaller values of  $\varepsilon$ ) periodicity in  $\Gamma^{1/3}$  is still the dominant feature but the shape of the bifurcation lines is distorted. Figure 13 shows the appearance of the lines  $\Sigma(\varepsilon, \gamma) = \text{const}$  in a  $\Delta/\ln \Gamma$  versus  $\Gamma^{1/3}$  plot (the horizontal line is  $\beta(\varepsilon, \gamma) = 1$ ). We now show that near  $\beta = 1$  the bifurcation structure of  $\mathbb{P}$  must reproduce that of  $\Pi$  if  $\Gamma$  is sufficiently large ( $\varepsilon$  sufficiently small).

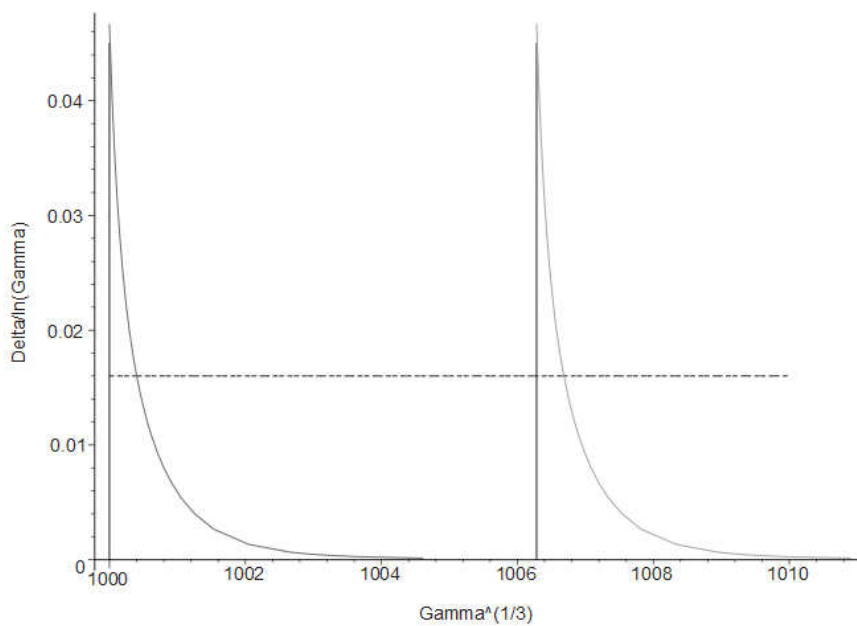


Fig. 13. The lines  $\Sigma = \text{const}$  after inclusion of the other terms in (233).



## 8.2. Inferences from $\Pi$ to $\mathbb{P}$

The mapping  $\mathcal{P}$ , Eq. (228) is equivalent to  $\mathbb{P}$  and may be written in components  $(\mathcal{P}_\Lambda, \mathcal{P}_\chi)$ :

$$\begin{aligned}\Lambda &\Rightarrow R_{L,f}(\varepsilon) + \varepsilon^{\kappa\pi} G(\Lambda, \chi, \varepsilon, \gamma) \\ &\equiv \mathcal{P}_\Lambda(\Lambda, \chi, \varepsilon, \gamma)\end{aligned}\quad (255a)$$

$$\begin{aligned}\chi &\Rightarrow \beta(\varepsilon, \gamma) \frac{\Lambda}{R_{L,f}(\varepsilon)} \cos(\chi + \tilde{\Sigma}(\Lambda, \varepsilon, \gamma)) \\ &\quad + \varepsilon^{\kappa\pi} H(\Lambda, \chi, \varepsilon, \gamma) \\ &\equiv \mathcal{P}_\chi(\Lambda, \chi, \varepsilon, \gamma)\end{aligned}\quad (255b)$$

$$\begin{aligned}\tilde{\Sigma}(\Lambda, \varepsilon, \gamma) &\equiv \Sigma(\varepsilon, \gamma) + \Phi_L(\Lambda, \varepsilon, \gamma) \\ &\quad - \Phi_L(R_{L,f}(\varepsilon), \varepsilon, \gamma)\end{aligned}\quad (255c)$$

where  $G$  and  $H$  are differentiable functions of  $\Lambda$  and  $\chi$ , bounded and with bounded derivatives (up to the third order) with respect to  $\Lambda, \chi$  as  $\varepsilon \rightarrow 0$  (uniformly with respect to  $\kappa$ , as long as  $\kappa > \kappa_0 > 0$ ). The difference  $\Phi_L(R_{L,f}) - \Phi_L(\Lambda)$  in (255c) is proportional to  $\Lambda^2 - R_{L,f}^2$  and thus, using (255a), to  $\varepsilon^{\kappa\pi}$ . It is convenient to regard in (255a), (255b) the quantity  $\varepsilon^{\kappa\pi}$  multiplying the functions  $G$  and  $H$  as an independent parameter  $q$  which may be set equal to zero (and obtain thus the mapping  $\Pi$ , Eq. (232) or to  $\varepsilon^{\kappa\pi}$  (to obtain  $\mathcal{P}$ ). With the assumptions concerning the functions  $G$  and  $H$  in (255a), (255b) the approach of  $\mathcal{P}$  to  $\Pi$  as  $q \rightarrow 0$  is uniform with respect to  $\chi, \Lambda, \beta \in [0, 2\pi] \times [0, \Lambda_M] \times [\beta_-, \beta_+]$ , where  $\Lambda_M$  is an upper bound on  $\Lambda$ , as derived in Sec. 4.1 and  $\beta_- < 1, \beta_+ > 1$  surround  $\beta = 1$ . As a consequence, we shall show generally that the properties of the circle map  $\Pi$  are carried over into those of the complete mapping  $\mathcal{P}$  for  $q$  sufficiently small, at fixed  $\varepsilon$ ; we may replace then the phrase “as  $q \rightarrow 0$ ” with the one “as  $\varepsilon \rightarrow 0$ ” using the approximate periodicity in  $\Sigma$  of  $\mathcal{P}_\chi$ , Eq. (255b). Indeed, if some property holds “for  $q < q_0$ ”, it will be true for all  $\varepsilon < q_0^{1/(\kappa\pi)}$ . In the process of letting  $q \rightarrow 0$  and choosing a correspondingly small  $\varepsilon$ , we assume first  $\beta = \text{const.}$  With this, it is easy to show:

**Lemma 8.1.** *If  $\chi_0$  is a fixed point of  $\Pi$ , Eq. (232), with  $d\Pi/d\chi(\chi_0) \neq 1$ , then, for sufficiently small  $\varepsilon$ ,  $\mathcal{P}$  also has a fixed point  $(\tilde{\Lambda}_0, \tilde{\chi}_0)$  so that:*

$$|\tilde{\Lambda}_0 - R_{L,f}|, |\tilde{\chi}_0 - \chi_0| = O(\varepsilon^{\kappa\pi}). \quad (256)$$

The argument — essentially the same as for the implicit function theorem — uses the periodicity of

$\Pi$  with respect to  $\Sigma$  and the fact that  $\mathcal{P}_\chi$  tends to  $\Pi$  as  $\varepsilon \rightarrow 0$ . First, for any  $\chi$  the equation  $\mathcal{P}_\Lambda(\Lambda, \chi, q) = \Lambda$  has a solution  $\Lambda = \Lambda(\chi, q)$  contained in  $(R_{L,f}(\varepsilon) - qM, R_{L,f}(\varepsilon) + qM)$  where  $M$  is an upper bound for  $|G(\Lambda, \chi)|$  of (255a). Indeed, the function  $\mathcal{P}_\Lambda(\Lambda, \chi) - \Lambda$  is monotonical with respect to  $\Lambda$  for small  $q$  and assumes opposite signs at the ends of the interval. The function  $\Lambda(\chi, q)$  is differentiable and

$$\frac{d\Lambda}{d\chi} = -\frac{\frac{\partial \mathcal{P}_\Lambda}{\partial \chi}}{\frac{\partial \mathcal{P}_\Lambda}{\partial \Lambda} - 1} = O(q). \quad (257)$$

Let now  $\delta > 0$  be such that:  $|d\Pi/d\chi(\chi_0) - 1| > \delta$  and  $\chi_-, \chi_+$  so that  $\chi_0 \in (\chi_-, \chi_+)$ ,

$$|\Pi(\chi_{+/-}) - \chi_{+/-}| > \delta \quad (258)$$

and

$$\left| \frac{d\Pi}{d\chi}(\chi) - 1 \right| > \delta, \quad \chi \in (\chi_-, \chi_+). \quad (259)$$

Using in  $\mathcal{P}_\chi(\Lambda, \chi, q)$  the function  $\Lambda(\chi, q)$  of (257), we evaluate (cf. (255c)):

$$\begin{aligned}&|\mathcal{P}_\chi(\chi, \Lambda(\chi)) - \Pi(\chi)| \\ &< \left| \beta \left( \frac{\Lambda}{R_{L,f}(\varepsilon)} - 1 \right) \cos(\chi + \tilde{\Sigma}) \right| \\ &\quad + |\beta \cos(\chi + \tilde{\Sigma}) - \beta \cos(\chi + \Sigma)| \\ &\quad + O(q).\end{aligned}\quad (260)$$

Both differences appearing on the right-hand side of (260) may be made as small as one wishes by allowing  $q$  to be sufficiently small. In particular, the left-hand side may become less than  $\delta/2$  with a choice of  $q$  valid uniformly with respect to  $\kappa$ , provided  $\kappa > \kappa_0 > 0$  and with respect to  $\chi \in [0, 2\pi]$ . Indeed, one estimates (cf. Eq. (255c)):

$$\begin{aligned}&\Phi_L(\Lambda) - \Phi_L(R_{L,f}(\varepsilon)) \\ &< \text{const} \times (\Lambda^2 - R_{L,f}^2) \frac{\beta}{R_{L,f} \mathcal{M}} \frac{1}{\kappa^{2/3} \left( \ln \left( \frac{1}{\varepsilon} \right) \right)^{2/3}} \\ &= O \left( \frac{q}{\kappa_0^{2/3}} \right).\end{aligned}\quad (261)$$

We conclude from (258) and (260) that the function  $\mathcal{P}_\chi(\Lambda(\chi), \chi, q) - \chi$  changes sign between  $\chi_-$  and  $\chi_+$ .

It is also monotonical there, if  $q$  is small enough. Indeed, the difference

$$\begin{aligned} & \left| \frac{d\mathcal{P}_\chi}{d\chi} - \frac{d\Pi}{d\chi} \right| \\ &= \left| \frac{d\Lambda}{d\chi} \right| \left| \frac{\mathcal{P}}{\Lambda} - \beta \frac{\Lambda}{R_{L,f}(\varepsilon)} \sin(\chi + \tilde{\Sigma}(\Lambda)) \frac{d\Phi_L}{d\Lambda} \right| \\ &+ \beta \frac{\Lambda}{R_{L,f}(\varepsilon)} |\sin(\chi + \tilde{\Sigma}(\Lambda)) - \sin(\chi + \Sigma(\Lambda))| \\ &+ O(q) \end{aligned} \quad (262)$$

may be rendered as small as one wishes, upon using (257), (261), in particular smaller than  $|1 - d\Pi/d\chi(\chi)|$ , cf. (259). Thus for  $q$  small enough,  $|d\mathcal{P}_\chi/d\chi - 1| \neq 0$ . It follows that, as announced,  $\mathcal{P}_\chi(\Lambda, \chi) - \chi$  vanishes just once between  $\chi_-$  and  $\chi_+$ . This proves Lemma 8.1.

Assume now  $\Pi(\chi)$  has several fixed points with  $\chi_i$  with  $d\Pi/d\chi(\chi_i) \neq 1$ . Choosing for every  $\chi_i$  corresponding intervals  $(\chi_-, \chi_+)$  as in (258), we may assume — using the uniformity of the approach of  $\mathcal{P}_\chi$  to  $\Pi$  — that (258) is valid at all points of  $[0, 2\pi]$  which lie outside the union of these intervals. We call this union  $I_\delta$ . Then, by letting  $q$  be small enough, we may ensure that:

$$\begin{aligned} |\mathcal{P}_\chi(\Lambda, \chi) - \chi| &> |\Pi(\chi) - \chi| - |\mathcal{P}_\chi(\Lambda, \chi) - \Pi(\chi)| \\ &> \frac{\delta}{2}, \quad (\Lambda, \chi) \in [0, \Lambda_M] \times \mathbb{C}I_\delta. \end{aligned} \quad (263)$$

Further,

$$\begin{aligned} |\mathcal{P}_\Lambda(\Lambda, \chi) - \Lambda| &> 0, \\ \chi \in I_\delta, \quad \Lambda \notin (R_{L,f} - qM, R_{L,f} + qM) \end{aligned} \quad (264)$$

as follows from the definition of  $M$ . But in each component of the remaining domain  $(R_{L,f} - qM, R_{L,f} + qM) \times I_\delta$ , according to the argument of Lemma 8.1 above, the equality:

$$|\mathcal{P}_\Lambda(\Lambda, \chi) - \Lambda| + |\mathcal{P}_\chi(\Lambda, \chi) - \chi| = 0$$

holds at only one point  $(\Lambda, \chi)$ , which is the “evolution” with  $q$  of the fixed point of the circle map  $\Pi$ . We conclude thus:

**Lemma 8.2.** *For  $\varepsilon$  sufficiently small, the fixed points of the complete mapping  $\mathcal{P}$  are in one-to-one correspondence with the fixed points  $\chi_i$  of the*

*circle map  $\Pi$ , provided the latter are such that  $|d\Pi/d\chi(\chi_i) - 1| > \delta$ , for some  $\delta > 0$ .*

Since only continuity arguments are involved, the same result is true for iterates of any order  $p$  of  $\mathcal{P}$ , when compared to the iterates  $\Pi^p$  of the circle map. Also, if  $\Pi(\chi)$  has an attracting/repelling fixed point with an eigenvalue of the linear part sufficiently far from unity, the eigenvalue of the linearization of  $\mathcal{P}$  around the corresponding fixed point is also smaller/larger than unity. It is easy to ascertain the stability of eigenvalues of  $\Pi$  equal to unity (saddle-node bifurcations) in the transition to the complete mapping  $\mathcal{P}$ , if one allows for a further degree of freedom: we take it to be the variable  $\beta$  of (229).<sup>26</sup> When varying  $\beta$ , we assume the quantity  $\Sigma$  is held fixed. This is only approximately the same as holding  $\varepsilon$  fixed: the departure is larger at smaller damping (see Fig. 13). One can state:

**Lemma 8.3.** *Let  $2\pi - \delta > |\Sigma + \pi/2| > \delta > 0$  and  $\chi_0, \beta_0$  so that  $\Pi(\chi) \equiv \beta_0 \Pi^0(\chi)$  (cf. Eq. (232),  $\Pi^0(\chi) \equiv \cos(\chi + \Sigma)$ ) obeys:*

$$\beta_0 \Pi^0(\chi_0) = \chi_0, \quad \beta_0 \frac{\partial \Pi^0}{\partial \chi}(\chi_0) = 1. \quad (265)$$

*Then, for  $q$  small enough, the set of equations*

$$\begin{aligned} \mathcal{P}_\Lambda(\Lambda, \chi, \beta, q) &= \Lambda, \\ \mathcal{P}_\chi(\Lambda, \chi, \beta, q) &= \chi, \quad \det[\mathbb{I} - \mathbb{D}\mathcal{P}] = 0 \end{aligned} \quad (266)$$

*has a unique solution  $(\Lambda(q), \chi(q), \beta(q))$  which tends to  $(R_{L,f}, \chi_0, \beta_0)$  as  $q \rightarrow 0$ .*

The argument is almost the same as in Lemma 8.1. In view of the continuity of  $\mathcal{P}_\chi$  and its derivatives as  $q \rightarrow 0$  it is true that: for any  $e > 0$  we can find  $q_0$  so that at fixed  $\Sigma$ , for  $0 < q < q_0$

$$\begin{aligned} |\mathcal{P}_\Lambda(\Lambda, \chi, \beta, q) - R_{L,f}| \\ + |\mathcal{P}_\chi(\Lambda, \chi, \beta, q) - \beta \Pi^0(\chi)| < e \end{aligned} \quad (267)$$

$$\left| \frac{\partial \mathcal{P}_\Lambda}{\partial \Lambda} \right| + \left| \frac{\partial \mathcal{P}_\Lambda}{\partial \chi} \right| + \left| \frac{\partial \mathcal{P}_\Lambda}{\partial \beta} \right| < e \quad (268)$$

and

$$\begin{aligned} \left| \frac{\partial \mathcal{P}_\chi}{\partial \chi} - \beta \frac{\partial \Pi^0}{\partial \chi} \right| < e, \quad \left| \frac{\partial^2 \mathcal{P}_\chi}{\partial \chi^2} - \beta \frac{\partial^2 \Pi^0}{\partial \chi^2} \right| < e \end{aligned} \quad (269)$$

<sup>26</sup>The  $\varepsilon$ -dependence of the factors  $R_{L,f}, M$  in (230) is very mild and may be overlooked for all purposes;  $\beta$  is essentially  $\varepsilon^{\kappa\pi/\gamma^{1/3}}$ .



for  $0 < \Lambda < \Lambda_M, \beta_- < \beta < \beta_+$  and all  $\chi$ . Further, the condition on  $\Sigma$  in the statement of the lemma ensures that  $\chi_0$  stays away from 0, i.e.  $\Pi(\chi) > d > 0$  in a domain  $(\chi_-, \chi_+) \times (\beta_-, \beta_+)$  containing  $(\chi_0, \beta_0)$ , for some  $d > 0$ . If  $e < d/2$ , it follows from (267) that  $|\mathcal{P}_{(\Lambda, \chi, \beta, q)}| > d/2 > 0$  on the same interval in  $\chi$  and  $\beta$ . As in Lemma 8.1, the first equation (266) determines for small  $q$  and all  $\chi \in [0, 2\pi]$  a unique solution  $\Lambda(\chi, \beta, q)$ , with  $\partial\Lambda/\partial\chi, \partial\Lambda/\partial\beta$  of  $O(q)$  and  $|\Lambda(\chi, \beta, q) - R_{L,f}| = O(q)$ . With the notation (similar to  $\Pi^0$  in (265), cf. Eq. (255b))

$$\mathcal{P}_\chi \equiv \beta \mathcal{P}_\chi^0 + qH(\Lambda, \chi, \beta, \Sigma) \quad (270)$$

the second equation (266) is:

$$\beta - \frac{\chi - qH(\Lambda(\chi, \beta, q), \chi, \beta, \Sigma)}{\mathcal{P}_\chi^0(\chi, \Lambda(\chi, \beta, q), q, \beta)} = 0. \quad (271)$$

In (271),  $\mathcal{P}_\chi^0$  depends on  $\beta$  also through the variable  $\tilde{\Sigma}$  (cf. Eqs. (261) and (255b)); from (261) it follows that  $\partial\tilde{\Sigma}/\partial\beta$  is  $O(\Lambda - R_{L,f})$ , i.e.  $O(q)$ . Since  $|\mathcal{P}_\chi| > d/2 > 0$  and  $\beta > \beta_-$ , it follows that the partial derivative of the left-hand side of (271) with respect to  $\beta$  is  $1 - O(q)$  so that (271) determines a function  $\beta(\chi, q)$  for  $\chi_- < \chi < \chi_+$ , with  $\partial\beta/\partial\chi = O(1)$ . The third equation (266) may be written after rearrangements:

$$\begin{aligned} \beta \left[ \frac{\partial \mathcal{P}_\chi^0}{\partial \chi} - \left( \frac{\partial \mathcal{P}_\chi^0}{\partial \chi} \frac{\partial \mathcal{P}_\Lambda}{\partial \Lambda} - \frac{\partial \mathcal{P}_\chi^0}{\partial \Lambda} \frac{\partial \mathcal{P}_\Lambda}{\partial \chi} \right) \right] \\ = 1 - q \left[ \frac{\partial H}{\partial \chi} \frac{\partial \mathcal{P}_\Lambda}{\partial \Lambda} - \frac{\partial H}{\partial \Lambda} \frac{\partial \mathcal{P}_\Lambda}{\partial \chi} \right] \end{aligned} \quad (272)$$

where we use  $\beta = \beta(\chi, q)$ . Since the derivatives of  $\mathcal{P}_\Lambda$  in (272) are  $O(q)$ , the latter may be rewritten as:

$$\beta \left[ \frac{\partial \mathcal{P}_\chi^0}{\partial \chi} + O(q) \right] = 1 + O(q). \quad (273)$$

From (271) and (273) we deduce:

$$\chi \frac{\partial \mathcal{P}_\chi^0}{\partial \chi} - \mathcal{P}_\chi^0 - O(q) = 0. \quad (274)$$

Now, at  $q = 0$  Eq. (274) has the solution  $\chi = \chi_0$ . This solution is a simple zero of the combination  $\chi \partial \Pi^0 / \partial \chi - \Pi^0$  because its derivative  $\chi \partial^2 \Pi^0 / \partial \chi^2$  is nonvanishing at  $\chi_0$ . This is a consequence of  $\partial^2 \Pi^0 / \partial \chi^2 = -\Pi^0$  and of the condition on  $\Sigma$ . Therefore, this combination is monotonical on an interval

$(\chi_-, \chi_+)$  around  $\chi_0$ , possibly included in the former, and acquires at the ends absolute values larger than some  $d > 0$ , possibly smaller than the former. Choosing again  $e < d/2$  we can find  $q_0$  so that, for  $q < q_0$  (i) the right-hand side of (274) has opposite signs at  $\chi_-, \chi_+$  and (ii) its derivative has a constant sign on  $(\chi_-, \chi_+)$  (as a consequence of the second equation in (269)). Thus there exists only one solution  $\chi(q)$  for every  $q$  sufficiently small and it approaches  $\chi_0$  as  $q \rightarrow 0$ . This leads then to solutions  $\Lambda(\chi(q)), \beta(\chi(q))$  with the properties announced in Lemma 8.3 and ends the argument.

Since now  $\beta$  changes with  $q$ , one may wonder about the values of the forcing  $\Gamma$  (or  $\varepsilon$ ) above (below) which we may set  $\varepsilon^{\kappa\pi} = q$  (and thus replace “small enough  $q$ ” with “small enough  $\varepsilon$ ”). According to Lemma 8.3, for  $q < q_0$ ,  $\beta(q)$  is contained in an interval  $[\beta_-, \beta_+]$ . For such values of  $\beta$ , the solutions  $\kappa(\varepsilon, \beta)$  of the equation  $\varepsilon^{\kappa\pi} / \gamma^{1/3} = \beta$  are at fixed  $\varepsilon$  contained in an interval  $[\kappa_-(\varepsilon), \kappa_+(\varepsilon)]$  which shrinks (logarithmically) to  $1/(8\pi)$  as  $\varepsilon \rightarrow 0$ . It is thus contained in an interval  $[\kappa_m, \kappa_M]$  of  $\kappa$ -values. It is enough to choose the upper limit of  $\varepsilon$  so that  $\varepsilon^{\kappa_M\pi} = q_0$ .

The invariant sets of  $\mathcal{P}$  in the neighborhood of  $(\Lambda(q), \chi(q), \beta(q))$  may be described completely using the central manifold theorem, in the manner presented in [Guckenheimer & Holmes, 1983] and [Marsden & McCracken, 1976]. Following the instructions of these references, we may state<sup>27</sup>:

**Lemma 8.4.** *Let  $\Sigma$  be such that:  $2\pi - \delta > |\Sigma + \pi/2| > \delta > 0$ . For  $q$  small enough, there exists a neighborhood  $U \times V$  of  $(\Lambda(q), \chi(q), \beta(q))$ ,*

$$U : \{|\Lambda - \Lambda(q)| < A\} \times \{|\chi - \chi(q)| < B\},$$

$$V : \{|\beta - \beta(q)| < C\}$$

*with  $A, B, C$  independent of  $q$  such that: if  $\beta < \beta(q)$ ,  $\beta \in V$ ,  $U$  contains no invariant sets of  $\mathcal{P}$ ; if  $\beta > \beta(q)$  the invariant set consists of two points; if  $\beta = \beta(q)$ , the only invariant set in  $U$  is  $(\Lambda(q), \chi(q))$ .*

To see this, we introduce new coordinates

$$\begin{aligned} \xi_1 &\equiv \Lambda - \Lambda(q), & \xi_2 &\equiv \chi - \chi(q), \\ \xi_3 &\equiv \beta - \beta(q) \end{aligned} \quad (275)$$

centered at  $\Xi_0 \equiv (\Lambda(q), \chi(q), \beta(q))$ . The mapping  $\mathcal{P}$  may be locally approximated by a Taylor

<sup>27</sup>The exercise on p. 25 of [Marsden & McCracken, 1976] is almost the same as this lemma.

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expansion:

$$\begin{aligned}\mathcal{P}_\Lambda : \xi_1 &\Rightarrow \sum_{i=1}^3 a_i \xi_i + \sum_{i \geq j=1}^3 a_{ij} \xi_i \xi_j + \cdots \\ \mathcal{P}_\chi : \xi_2 &\Rightarrow \xi_3 \mathcal{P}_\chi^0(\Xi_0) + b_1 \xi_1 + \left(1 - \frac{a_2 b_1}{1 - a_1}\right) \xi_2 \\ &+ \sum_{i \geq j=1}^3 b_{ij} \xi_i \xi_j + \cdots\end{aligned}\quad (276)$$

where  $a_1, a_2$  are  $O(q)$ ,  $a_3$  even of  $O(q^2)$  (cf. Eq. (224a)), where the dependence of  $\beta$  is concealed in  $\Sigma$ , (cf. also (255c)),  $a_{ij}$  are all of  $O(q)$ , but nonvanishing (cf. Eq. (224a)),  $\mathcal{P}_\chi^0(\Xi_0) \approx \Pi^0(\chi_0) \neq 0$ , the coefficient of  $\xi_2$  is  $1 - O(q)$  and  $b_{ij}$  are nonvanishing, as a consequence of the condition on  $\Sigma$ , and of  $O(1)$ . For small  $q$ , the coefficient  $b_{22}$  is approximately  $\partial^2 \Pi^0 / \partial \chi^2 \neq 0$ . Let then  $\xi'_1, \xi'_2$  be new coordinates linearly related to  $\xi_1, \xi_2$  so that the linear part in  $\xi_1, \xi_2$  of (276) becomes diagonal. The first eigenvalue is  $O(q)$ ; it is relevant that  $\xi'_1 = O(1)\xi_1 + O(q)\xi_2$  so that the orders of magnitude in the transformed system:

$$\begin{aligned}\mathcal{P}_1 : \xi'_1 &\Rightarrow c_1 \xi'_1 + a'_3 \xi'_3 + \sum_{i \geq j=1}^3 a'_{ij} \xi'_i \xi'_j \\ \mathcal{P}_2 : \xi'_2 &\Rightarrow \xi'_3 (\mathcal{P}_\chi^0(\Xi_0) + O(q)) + \xi'_2 \\ &+ \sum_{i \geq j=1}^3 b'_{ij} \xi'_i \xi'_j + \cdots\end{aligned}\quad (277)$$

are preserved. In (277) we have set  $\xi'_3 = \xi_3$ . We enlarge  $\mathcal{P}$  by adding to (277):

$$\mathcal{P}_3 : \xi'_3 \Rightarrow \xi'_3. \quad (278)$$

The quantity  $a'_3$  is  $O(q)$ ,  $b'_{22} \approx b_{22}$  and  $b'_{ij}$  are  $O(1)$ . It is possible to find an approximation to the central manifold of the enlarged  $\mathcal{P}$  (Eqs. (277), (278)) (the invariant manifold tangent to the subspace of eigenvalue unity) around  $\Xi_0$  in the form (see [Guckenheimer & Holmes, 1983, p. 136])

$$\begin{aligned}\xi'_1 &= h_I(\xi'_2, \xi'_3) \\ &= \alpha_{22}(\xi'_2)^2 + \alpha_{23}\xi'_2 \xi'_3 \\ &+ \alpha_{33}(\xi'_3)^2 + \cdots\end{aligned}\quad (279)$$

The coefficients  $\alpha_{ij}$  are obtained by equating the coefficients of like powers of  $\xi'_2, \xi'_3$  in the condition:

$$\begin{aligned}\mathcal{P}_1(h_I(\xi'_2, \xi'_3), \xi'_2, \xi'_3) \\ = h_I(\mathcal{P}_2(h_I(\xi'_2, \xi'_3), \xi'_2, \xi'_3), \xi'_2, \xi'_3).\end{aligned}\quad (280)$$

It turns out that  $\alpha_{ij}$  are all of  $O(q)$ , as they are proportional to coefficients appearing in  $\mathcal{P}_1$ . According to the center manifold theorem (see [Marsden & McCracken, 1976, p. 19]) all points in a (sufficiently small) neighborhood  $U \times V$  of  $\Xi_0$  approach under iterations of  $\mathcal{P}$  the center manifold, which is itself invariant. Thus the only possible invariant sets of  $\mathcal{P}$  are to be found by restricting the action of  $\mathcal{P}$  to it. Substituting then Eq. (279) into the second equation (277) we obtain a description of the bifurcations at  $\Xi_0$ :

$$\begin{aligned}\hat{\mathcal{P}}_2(\xi'_2, \xi'_3) : \xi'_2 &\Rightarrow \xi'_3 (\cos(\chi(q) + \Sigma) + O(q)) \\ &+ \xi'_2 (1 + b'_{23} \xi'_3) + b'_{22} (\xi'_2)^2 \\ &+ O(q) \times \text{h.o.}\end{aligned}\quad (281a)$$

$$\hat{\mathcal{P}}_3(\xi'_3) : \xi'_3 \Rightarrow \xi'_3. \quad (281b)$$

For  $q$  small enough, Eq. (281a) describes a saddle-node bifurcation: for  $\xi'_3 > 0$ , the equation  $\xi'_2 = \hat{\mathcal{P}}_2(\xi'_2, \xi'_3)$ , with  $\hat{\mathcal{P}}_2$  restricted to the quadratic part has no solutions for small  $\xi'_3$ ; it has one double zero for  $\xi'_3 = 0$  and two solutions for  $\xi'_3 < 0$ . Thus saddle-node bifurcations of the circle map are transferred indeed to the complete mapping  $\mathcal{P}$  at least for small  $q$ , i.e. for small  $\varepsilon$ .

We show that the same is true for flip bifurcations. In strict analogy to Lemma 8.3 it is true that:

**Lemma 8.5.** *Let  $2\pi - \delta > |\Sigma - \pi/2| > \delta > 0$  and  $\chi_0, \beta_0$  obeying (cf. Lemma 8.3):*

$$\beta_0 \Pi_0(\chi_0) = \chi_0, \quad \beta_0 \frac{\partial \Pi_0}{\partial \chi}(\chi_0) = -1. \quad (282)$$

*Then, for  $q$  small enough, the set of equations:*

$$\begin{aligned}\mathcal{P}_\Lambda(\Lambda, \chi, q) &= \Lambda, \\ \mathcal{P}_\chi(\Lambda, \chi, q) &= \chi, \quad \det[\mathbb{I} + \mathbb{D}\mathcal{P}] = 0\end{aligned}\quad (283)$$

*has a unique solution  $(\Lambda(q), \chi(q), \beta(q))$  which tends to  $(R_{L,f}, \chi_0, \beta_0)$  as  $q \rightarrow 0$ .*

The argument is the same as in Lemma 8.3 with obvious changes of sign in (273) and (274). The same transformations of variables as in Lemma 8.4 bring  $\mathcal{P}$  locally in the form (277) with a minus sign in front of  $\xi'_2$  in the linear part of  $\mathcal{P}_2$ . We inquire next whether the flip bifurcations survive the transition from one to two dimensions (from  $\Pi$  to  $\mathcal{P}$ ):



**Lemma 8.6.** *Let  $\Sigma$  obey:  $2\pi - \delta > |\Sigma - \pi/2| > \delta > 0$ . For  $q$  small enough, there exists a neighborhood  $U \times V$  of  $(\Lambda(q), \chi(q), \beta(q))$ ,  $U : \{|\Lambda - \Lambda(q)| < A\} \times \{|\chi - \chi(q)| < B\}$ ,  $V : \{|\beta - \beta(q)| < C\}$ , with  $A, B, C$  independent of  $q$  and such that: if  $\beta \leq \beta(q)$ ,  $\beta \in V$ ,  $U$  contains just one invariant set of  $\mathcal{P} \circ \mathcal{P}$ , which is also an invariant set of  $\mathcal{P}$ ; if  $\beta > \beta(q)$  the invariant set of  $\mathcal{P}$  consists of one fixed point and one stable orbit of period two.*

One may doubt *a priori* that this is the case, since pitchfork bifurcations (as present in the mapping  $\Pi \circ \Pi$ ) are not stable under perturbations in general. Nevertheless, the special features appearing from the restriction to mappings of the form  $\mathcal{P} \circ \mathcal{P}$  allow a proof of the persistence of flip bifurcations when one moves from  $\Pi$  to  $\mathcal{P}$ .

Expanding  $\mathcal{P}$  around the point  $(\Lambda(q), \chi(q), \beta(q))$  of Lemma 8.5 the mapping  $\mathcal{P} \circ \mathcal{P}$  may be written with the notations of (275):

$$\begin{aligned} \mathcal{P}_1^2 : \xi_1 &\Rightarrow a_1 \xi_3 + a_2 \xi_1 + a_{11} \xi_1^2 + a_{22} \xi_2^2 \\ &\quad + a_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 + a_{33} \xi_3^2 + \dots \\ \mathcal{P}_2^2 : \xi_2 &\Rightarrow \xi_2 + b_{11} \xi_1^2 + b_{12} \xi_1 \xi_2 + b_{13} \xi_1 \xi_3 \\ &\quad + b_{23} \xi_2 \xi_3 + b_{33} \xi_3^2 + d_{222} \xi_2^3 \\ &\quad + d_{223} \xi_2^2 \xi_3 + d_{233} \xi_2 \xi_3^2 + \dots \\ \mathcal{P}_3^2 : \xi_3 &\Rightarrow \xi_3 \end{aligned} \quad (284)$$

where  $a_i, a_{ij}$  are of  $O(q)$  or less,  $a_2 > 0$ ,  $a_{12} = 0$ ,  $b_{ij}$  are of  $O(1)$ ,  $b_{22} = 0$ ,  $b_{23} \neq 0$  (as  $q \rightarrow 0$  it approaches  $-\partial^2 \Pi / \partial \beta \partial \chi = 1/\beta_0$ ) and terms of  $O(\xi^3)$  must be taken into account (to describe the pitchfork); the coefficient  $d_{222} \neq 0$  and has a nonzero limit as  $q \rightarrow 0$ , as a consequence of the conditions on  $\Sigma$ . In (284) we extended  $\mathcal{P}^2$  through the addition of the identity concerning  $\xi_3$  (see [Guckenheimer & Holmes, 1983]). Changing variables to:

$$\xi_1 = \xi'_1 + \frac{a_1}{1 - a_2} \xi_3, \quad \xi'_2 = \xi_2, \quad \xi'_3 = \xi_3 \quad (285)$$

causes the linear term  $a_1 \xi_3$  to disappear in the transformed equations. One looks for an invariant manifold of  $\mathcal{P}^2$  tangent to the plane  $\xi'_1 = 0$  in the form (279) above. From an equation analogous to (280) one determines the coefficients  $\alpha_{ij}$  which turn out to be of  $O(q)$  or smaller. Replacing  $\xi_1$  as a function of  $\xi_2, \xi_3$  in the second equation (284) one obtains a description of the invariant sets in the approximation (279) of the center manifold of the

extended system (284):

$$\begin{aligned} B_{23} \xi_2 \xi_3 + B_{33} \xi_3^2 + B_{222} \xi_2^3 + \xi_3 O(\xi_2^2, \xi_2 \xi_3, \xi_3^2) \\ + O(q)(\xi_2^4 + \dots) = 0 \end{aligned} \quad (286)$$

where  $B_{23}, B_{33}$  are corrections of  $O(q)$  to  $b_{23}, b_{33}$  of (284) and are of  $O(1)$ ,  $B_{222}$  is a correction of  $O(q)$  to  $d_{222}$  in (284), the magnitude of the coefficients of the other cubic terms is  $O(1)$ ; it is essential that no terms with  $\xi_2^2$  appear, even of  $O(q)$ : these would destroy the “pitchfork”; there are terms of  $O(\xi_2^4)$  multiplied by  $O(q)$ , but these are harmless. The prime on  $\xi_2, \xi_3$  has been dropped in (286), in view of (285). At  $\xi_3 = 0$  (286) reduces to:

$$B_{222} \xi_2^3 + O(q) \xi_2^4 = 0 \quad (287)$$

which shows the triple zero at  $\xi_2 = 0$  and another zero far away. A solution of (286) which is analytic in  $\xi_3$  may be obtained formally by writing:

$$\xi_2 = e_1 \xi_3 + e_2 \xi_3^2 + \dots \quad (288)$$

and identifying coefficients; one obtains  $e_1 = -B_{33}/B_{23}$ , etc. There exist also two other solutions analytic in  $\sqrt{\xi_3}$  which are the continuations in  $\xi_3$  of the other zeroes of (287): letting in (286)  $\xi_3 = x^2$  and substituting there:

$$\xi_2 = f_1 x + f_2 x^2 + \dots \quad (289)$$

one determines  $f_1$  as one solution of:

$$B_{23} f_1 + B_{222} f_1^3 = 0. \quad (290)$$

The solution  $f_1 = 0$  leads to (288). The coefficients  $B_{23}, B_{222}$  are corrections of  $O(q)$  to the derivatives  $\partial^2 \Pi / \partial \beta \partial \chi, \partial^3 \Pi / \partial \chi^3$  at  $(\chi_0, \beta_0)$ : these are  $\approx -1/\beta_0, 1$  in turn. Thus,  $f_1 \approx \pm \beta_0^{-1/2}$ . This shows that indeed, for  $\xi_3 < 0$  there exists just one solution of (286) but there are three solutions for  $\xi_3 > 0$ , i.e. a “pitchfork”. This ends the argument for Lemma 8.6.

### 8.3. Conclusions

In the statements of Lemmas 8.3–8.6, the values of  $\Sigma$  lying near the peaks of the saddle-node and flip bifurcation curves (see Fig. 11) were excluded. With this exception, we can conclude this section with the following.

**Theorem 8.1.** *If  $\beta > \beta_e(\Sigma)$  of Lemma 7.2, the invariant sets of the half-period Poincaré map  $\mathbb{P}(\beta, \Sigma, \Lambda, \chi)$  of Duffing’s equation consist, for sufficiently large  $\Gamma$  of fixed points and periodic points*

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of period two only — with the possible exception of small neighborhoods, vanishing as  $\Gamma$  increases, of points ( $\beta = 1, \Sigma = \pm\pi/2$ ). These invariant points are in one-to-one correspondence with those of the circle map  $\Pi(\beta, \chi, \Sigma)$  of (232) and approach the latter as  $\Gamma \rightarrow \infty$  ( $\varepsilon \rightarrow 0$ ). The bifurcation lines  $\beta_F(\Sigma)$ ,  $\beta_S(\Sigma)$  approach those of  $\Pi$  in this limit.

The restriction to the domain of large  $\beta$  is artificial: one can extend the argument and show the stability in the transition  $\Pi \Rightarrow \mathcal{P}(\equiv \mathbb{P})$  of the period doubling cascade and — presumably — of the limiting chaotic motion. This gives a natural explanation for the chaotic behavior observed a long time ago in the damped and forced Duffing oscillators. Theorem 8.1 also gives a complete understanding of the regularities observed in the bifurcation pattern of Duffing's equation at large forcing and (moderately) high damping.

## 9. Comments and Conclusions

It is apparent that an important ingredient in the justification of the bifurcation structure of Eq. (1) (or Eq. (8)) is the sudden change of “natural” reference at  $t = 0$  in the description of the motion. This is brought about by the discontinuity occurring at  $t = 0$  when passing from the left-hand reference  $X_L(t)$  to  $X_R(t)$  (see Fig. 2). The continuation of  $X_L(t)$  to  $t > 0$  is oscillatory (see Fig. 6) because (intuitively) a particle moving for  $t < 0$  at the bottom of the potential well  $x^4/4 - x \sin t$  (cf. Eq. (8)) cannot follow the infinite velocity of the minimum at  $t = 0$ . The particle behaves as if it had been subjected to a “kick”. In a series of papers by Parlitz [1993] and Parlitz *et al.* [1991a, 1991b], the authors showed that an infinite sequence of bifurcation curves occur in a very simple “kick and twist model”, described by the differential equations ( $x = r \cos \alpha, y = r \sin \alpha, d > 0$ ):

$$\dot{r} = -\frac{d}{2}r \quad \dot{\alpha} = 1 + r^2 \quad (291)$$

supplemented by a periodic “kick”, i.e. a displacement of the  $y$  coordinate by an amount  $a$  at equally spaced time intervals  $T$ . The control parameters are amplitude  $a$  and period  $T$ .

For Duffing-type equations Eilenberger and Schmidt [1992] gave a simple and elegant argument that for a *real* sudden change of the forcing at  $t = 0$  (obtained by replacing  $\sin t$  on the right-hand side of (8) by a step function) a nonlinear

dependence of the restoring force on the displacement  $x(t)$  leads naturally at sufficiently high damping to a (half-period) Poincaré map of the form (4) or (232), i.e. to a “circle map”. The latter describes accurately the bifurcation structure of the Duffing equation at high forcing and damping. The critique to this argument is obviously that there is in (8) or (9) no real discontinuous change of forcing (no real “kick” [Eilenberger & Schmidt, 1992]) at  $t = 0$ : the change of reference is only a convenient artefact. As is apparent from Sec. 6, see Fig. 8 or Fig. 9, the development of the circle map occurs actually in a short time interval  $\tau = O(1/\gamma)$  near  $t = 0$ . In the limit  $\Gamma \rightarrow \infty$  this interval becomes infinitely short compared to the whole interval  $[0, \pi/2]$  (actually  $O(1/\ln(1/\varepsilon))$  compared to it) but is infinitely long ( $O(1/(\varepsilon^{3/8} \ln 1/\varepsilon))$ ) compared to the boundary layer, where  $\tau$  (cf. Eq. (18)) is  $O(1)$  (this is the — somewhat enlarged — *transition* region of Schmidt and Eilenberger [1998] where the inner and outer expansions are *matched*, see Sec. 3).

The following is a qualitative argument for the appearance of the circle map as an approximation to the half-period Poincaré map — relating  $t = -\pi/2$  to  $t = \pi/2$ , as it emerges from the discussion in Secs. 5 and 6. The discussion ignores the difficulties related to the changes of variable between  $\tau$  and  $\theta_{L,R}$  of (84) and assumes they can be performed all the way down to  $t = 0$ . In the time interval  $[-\pi/2, -\tau_0 \varepsilon^{3/8-s}]$ ,  $s > 0$  the motion around the left-hand reference solution is essentially harmonic in the variable  $\theta_L$ , Eq. (84) i.e. with a period independent of the amplitude. According to Lemma 5.2 there exists an additional phase proportional to the square of the amplitude at  $t = -\pi/2$  which is a cause for the distortions shown in Fig. 5. To these, we have to add the displacements appearing when we get into the boundary layer down to  $t = 0$  where the motion of solutions in the vicinity of  $X_L(t)$  is approximately described by the linear equation (153). Figure 5 testifies however that the distortion of the small disk at  $t = -\pi/2$  in the  $t = 0$  plane is not a huge effect, so that we make the rough approximation that harmonic motion is dominant and thus all *initial* phases  $\psi_i$  — measured around  $X_L(t)$  at  $t = -\pi/2$  — have increased at  $t = 0$  by the same amount  $a$ , independent of the amplitude:

$$\psi(0) = \psi_i + a. \quad (292)$$

At  $t > 0$  the motion consists essentially of rotations around the right-hand reference solution



$X_R(t)$ . The angular velocity has an harmonic (amplitude-independent) term in the variable  $\theta_R$ , Eq. (84) and (to a first approximation) a second term which decreases with  $\theta_R$  like  $1/\theta_R^{3/4}$  and is proportional to the square of the amplitude (of the distance from the origin in the  $w, dw/d\theta$  plane, see Eqs. (182a), (182b)). This latter term is the prominent effect of the nonlinearities in (1). Its effect is limited to a time  $\tau < 1/\gamma$ , after which the motion is essentially harmonic. The final value  $\psi_f$  of the angle at  $t = \pi/2$  is the value of the rotation angle around  $X_R(t)$  between  $t = 0$  and  $\pi/2$ . Thus, for a solution rotating at distance  $R$  from  $X_R(t)$  (with  $R$  measured in units of  $\varepsilon^{1/8}$ ),

$$\psi_f = \frac{A}{\sqrt{\varepsilon}} + \frac{BR^2}{\gamma^{1/3}} + C. \quad (293)$$

The first term is common to all solutions and is the effect of the harmonic part:  $1/\sqrt{\varepsilon}$  is the order of magnitude of  $t = \pi/2$  when expressed in the variable  $\theta_R$ , Eq. (84). The second term is the integral over  $\theta_R^{-3/4}$  up to  $\tau = 1/\gamma$ ; since  $\theta_R \approx \tau^{4/3}$  a time  $\tau = 1/\gamma$  corresponds to  $\theta_R \approx 1/\gamma^{4/3}$ . The third term is the contribution to the rotation angle of times larger than  $1/\gamma$  and is again independent of the chosen solution.

The continuation to  $t > 0$  of the left-hand reference solution  $X_L(t)$  also rotates around  $X_R(t)$  at a distance  $R_L$  from it (cf. Fig. 6) and achieves at  $t = \pi/2$  a total rotation (293) given by an angle denoted by  $\psi_{f,L}$ . Neighboring solutions with a distance  $R_L + \delta R$  to  $X_R(t)$  acquire a rotation angle:

$$\psi_f(\delta R) = \psi_{f,L} + \frac{2BR_L\delta R}{\gamma^{1/3}}. \quad (294)$$

We assume  $R_L$  is constant down to  $t = 0$ . A solution starting at  $t = -\pi/2$  close enough (i.e.  $O(\varepsilon^{3/16+\kappa\pi/2})$ , cf. Theorem 4.1) to  $X_L(t)$  with an angle  $\psi_i$  to the  $x$ -axis ends up at  $t = 0$  in a disk of radius  $r = O(\varepsilon^{\kappa\pi+1/8}) = O(e^{-\Delta\pi}\varepsilon^{1/8})$  around  $X_L$  (cf. Eqs. (166a), (166b)) and in a position enclosing an angle  $\psi(0)$ , Eq. (292) with the  $x$ -axis. Its distance to  $X_R(0)$  is:

$$\begin{aligned} R_L + \delta R &= \sqrt{R_L^2 + r^2 + 2R_L r \cos(\psi + a)} \\ &\approx R_L + r \cos(\psi_i + a) \end{aligned} \quad (295)$$

which gives an estimate of  $\delta R$ . Substituting in (294) one obtains the circle map, which gives the angle  $\psi_f$  at  $t = \pi/2$  in terms of  $\psi_i$  at  $t = -\pi/2$ :

$$\psi_f = \psi_{f,L} + \frac{\text{const} \times \varepsilon^{\kappa\pi}}{\gamma^{1/3}} \cos(\psi_i + a). \quad (296)$$

This mapping is equivalent to the form (4) or (232) (cf. Sec. 7.1). The entire paper is actually devoted to the justification of this qualitative picture in a correct manner.

The role of the nonlinearities may be appreciated if one compares Eq. (8) with a (possible) linear version of it:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + x = (\sin t)^{1/3} \quad (297)$$

which has, for all  $\varepsilon, \mu$  a unique periodic solution. One can perform the same analysis with inner and outer expansions as for (8).<sup>28</sup> The difference  $u(t) \equiv x(t) - x_L(t)$  to a corresponding reference solution  $x_L(t)$  obeys the equation of a linear damped harmonic oscillator; the latter transforms a small disk  $u^2 + \dot{u}^2/\varepsilon < r^2$  at  $t = -\pi/2$  into a disk of radius smaller by a factor  $\varepsilon^{\kappa\pi}$  at  $t = \pi/2$  (around a corresponding reference  $x_R(t)$ ), almost without change of shape (to first order in  $\mu/\sqrt{\varepsilon}$ ). The difference in rotation angles for different amplitudes, as expressed by (294) in the nonlinear case, is zero.

The author believes it is a special virtue of the averaging method of Bogolyubov and Mitropolski [1961] that it allows a systematic and easily interpretable treatment of the nonharmonic behavior in the small  $|t|$  domain (the “transition region” of [Schmidt & Eilenberger, 1998]). In fact, use of this method makes up the main difference between the treatment of my earlier internal report [Stefanescu, 1990] and of the present work to that of the papers of Eilenberger and Schmidt [1992], Schmidt and Eilenberger [1998]. The analysis of these authors is based on the adiabatic theorem of classical mechanics [Arnold, 1978; Landau & Lifshitz, 1960], applied to the motion described by Eq. (9) around the reference (“creeping”) solutions  $X_L(t\sqrt{\varepsilon}), X_R(t\sqrt{\varepsilon})$ .<sup>29</sup> It is not so easy to extend the adiabatic approximation to the region of small  $|t|$  in such a manner that it matches the boundary layer description of the motion, given to zeroth order by Eq. (64a).

<sup>28</sup>Inner variables  $X, \tau$  may be defined through  $x \equiv \varepsilon^{1/6} X, t = \varepsilon^{1/2} \tau$ .

<sup>29</sup>The latter are introduced in a manner similar to this paper and to the work of Byatt-Smith [1987].



This is done by a method of time-dependent canonical transformations, specially devised for this purpose and presented in Appendix B of [Schmidt & Eilenberger, 1998]. The procedure is claimed to be numerically successful but it is difficult to identify the various terms given by the averaging method of Secs. 5.1 and 6.

Both the work of Eilenberger and Schmidt [1992, 1998] and the present one (with its earlier version) rely on a Taylor expansion to first (or second order) of the Poincaré map around the continuation of the left-hand reference solution to  $t > 0$ . This statement may not be obviously valid for the work of Eilenberger and Schmidt because of the different formulation of the Poincaré map, but a moment's consideration shows that it is implied in Eqs. (122), (123) of [Eilenberger & Schmidt, 1992] and in Eq. (24) of [Schmidt & Eilenberger, 1998]. The reason why a Taylor expansion is expected to be sufficient is that the magnitude of the small disk of radius  $\varepsilon^{\kappa\pi/2+3/16}$  is further reduced by a factor<sup>30</sup>  $\varepsilon^{\kappa\pi/2}$  at  $t = 0$ , so that one is interested in a “really” small neighborhood of the continuation of  $X_L(t)$  at  $t > 0$ .

For given  $\Delta$  and  $\Gamma$  in the large  $\Delta - \Gamma$  region considered here (or  $\varepsilon$  and  $\mu$  in the corresponding domain) one determines numerically the coefficients of this expansion (as in [Schmidt & Eilenberger, 1998]) or integrates the variational equation around  $X_L(t)$  from  $t = 0$  to  $t = \pi/2$ , as analyzed (in principle) in this work (and in the earlier report); the advantage of the latter method is that the averaging transformations allow several statements about the solutions (see Sec. 6.2), especially about their behavior as  $\varepsilon \rightarrow 0$ . Really “universal” numerical constants appear only in this limit (cf. Eqs. (212), (213)). They are determined by the boundary layer equations in the limit  $\gamma \rightarrow 0$ . This paper devotes much attention to the justification of these limits (see Lemma 6.4), the reason being that not all of them *exist*: see Eqs. (222), (223). It is in fact the equilibrium between this divergence and the magnitude of the small disk at  $t = 0$  (see Fig. 5) which is responsible for the onset of bifurcations as the damping decreases at fixed  $\Gamma$ . Unfortunately, a similar discussion appears to be absent in the work of Eilenberger and Schmidt.

It is remarkable that the circle map — which comes from the *first* order Taylor expansion — turns out, according to the numerical evidence of Schmidt and Eilenberger [1998], to have a large domain of validity — at least qualitatively — in the  $\Gamma - \Delta$  plane, not limited to values of the damping increasing logarithmically with the forcing (as assumed in this paper). It is an open question to what extent (down to which value of the damping  $\Delta$ ) the very rich bifurcation structure of the circle map is indeed transferred to the highly complex bifurcation diagram of the Duffing equation in the  $\Gamma - \Delta$  plane. The arguments of Sec. 8.1 only show that the transfer of the upper part of the bifurcation curves does occur asymptotically in  $\Gamma$ .

## Acknowledgments

This paper — which is a new formulation of my internal report [Stefanescu, 1990] — originated in discussions with G. Höhler more than twenty years ago. He was engaged at that time on a numerical study of the bifurcation diagram of the Duffing equation [Höhler, 1993]. A (very) short exchange of e-mails with J. Gallas last year determined me to reconsider my work of that time and look at it carefully again. I wish to thank him for this correspondence.

All calculations (numeric and symbolic) were performed with a MAPLE10 program.

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<sup>30</sup>The factor  $\varepsilon^{3/16}$  disappears through rescaling and Liouville transformation.



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## Appendices

### A. Bound on the Increase of $E(\cdot)$ in the Interval $(\cdot)^{3p; \cdot q}$

In this Appendix we drop the index  $R$  on  $\theta_R$ ,  $w_R$ ,  $\theta_{0R}$ ,  $g_R$  because the variables for the interval  $(-\pi/2, 0)$  do not occur at all. The change of

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variables  $W = wk(\theta)$  transforms Eq. (95) into:

$$\frac{d^2W}{d\theta^2} - \frac{2}{k^2} \frac{dk}{d\theta} \frac{dW}{d\theta} + W(1 + G(\theta)) + W^2 + \frac{W^3}{3} = 0 \quad (\text{A.1})$$

where

$$G(\theta) \equiv g(\theta) + \frac{2}{k^2} \left( \frac{dk}{d\theta} \right)^2 - \frac{1}{k} \frac{d^2k}{d\theta^2}. \quad (\text{A.2})$$

This function decreases like  $1/\theta^2$  for large  $\theta$ , independently of the choice of  $p$ , Eq. (93) in Sec. 4.2. The energy associated to (A.1):

$$\begin{aligned} \mathcal{E}(\theta) &= \frac{1}{2} \left( \frac{dW}{d\theta} \right)^2 + \frac{W^2}{2} (1 + G(\theta)) \\ &\quad + \frac{W^3}{3} + \frac{W^4}{12} \end{aligned} \quad (\text{A.3})$$

evolves in time according to:

$$\frac{d\mathcal{E}}{d\theta} = \frac{W^2}{2} \frac{dG}{d\theta} - \frac{3}{4\theta} \left( \frac{dW}{d\theta} \right)^2 \quad (\text{A.4})$$

where we have used the approximation  $k(\theta) \approx \varepsilon^{3p/2}/t^{1/2}$  for small  $t$ . The energy  $\mathcal{E}$  is bounded because, using  $W^2 < 6\mathcal{E}$  (cf. Eq. (98)), Eq. (A.4) leads to the inequality:

$$\frac{d \ln \mathcal{E}}{d\theta} < \text{const} \left| \frac{dG}{d\theta} \right| \quad (\text{A.5})$$

which means  $\mathcal{E} < \text{const} \mathcal{E}(\theta_0)$ .<sup>31</sup> This is still a very weak bound for  $w(\theta)$ : it simply implies that  $|w(\theta)| < \text{const}/k(\theta)$  i.e. using (105)

$$\begin{aligned} |w(\theta)| &< \text{const} \sqrt{\mathcal{E}(\theta)} \left( \frac{\theta}{\theta_0} \right)^{3/8} \\ &< \text{const} \sqrt{\mathcal{E}(\theta_0)} \left( \frac{\theta}{\theta_0} \right)^{3/8}. \end{aligned} \quad (\text{A.6})$$

We consider now in more detail the negative “damping” term in (A.4): by comparing with the motion  $W_f(\theta)$  in the time independent potential:

$$V(W) = \frac{W^2}{2} + \frac{W^3}{3} + \frac{W^4}{12} \quad (\text{A.7})$$

we shall show that, in fact, the energy  $\mathcal{E}$  decreases to zero like  $\text{const}/\theta^s$ , for some  $s > 0$ . Let  $T(\mathcal{E})$  be the

period of the motion with energy  $\mathcal{E}$  in the potential  $V(W)$ , Eq. (A.7) and let  $T_M \equiv \sup T(\mathcal{E})$  over all  $\mathcal{E} > 0$ .<sup>32</sup> Further let  $T$  be a time interval obeying  $T > qT_M$  for some integer  $q > 1$ . The following is useful:

**Statement A.1.** *There exists a constant  $k_T$ ,  $0 < k_T < 1$  so that the inequality:*

$$\left( \frac{dW}{d\theta} \right)^2 > \mathcal{E} \quad (\text{A.8})$$

*is fulfilled by the motion  $W_f(\theta)$  with energy  $\mathcal{E}$  in the fixed potential  $V(W)$ , Eq. (A.7) during a time  $k_T T$ , independently of the energy of the motion.*

Indeed, we evaluate first the fraction  $\tilde{k}(\mathcal{E})$  of a period of the motion with energy  $\mathcal{E}$  during which (A.8) is obeyed. Let first  $\mathcal{E} > 1$ :

$$\tilde{k}(\mathcal{E}) = \frac{I\left(\frac{\mathcal{E}}{2}\right)}{I(\mathcal{E})} \quad (\text{A.9})$$

with

$$\begin{aligned} I(f\mathcal{E}) &\equiv \int_{W(-,f\mathcal{E})}^{W(+,f\mathcal{E})} \frac{dW}{\sqrt{\mathcal{E} - V(W)}} \\ &= \frac{1}{\mathcal{E}^{1/4}} \int_{u(-,\lambda)}^{u(+,\lambda)} \frac{du}{\sqrt{1 - \frac{\lambda^2 u^2}{2} - \frac{\lambda u^3}{3} - \frac{u^4}{12}}} \end{aligned}$$

where  $W(-/+, \mathcal{E})$  are the two real roots of the equation  $V(W) = f\mathcal{E}$ ,  $u(-/+, \lambda)$  are the roots of the corresponding equation after changing variables to  $W = u\mathcal{E}^{1/4}$ ,  $f = 1/2, 1$  and  $\lambda = 1/\mathcal{E}^{1/4}$ . The function  $\tilde{k}(\mathcal{E}(\lambda))$  is a continuous, strictly positive function of  $\lambda$  on the closed interval  $[0, 1]$  and achieves there its smallest value which is different from zero (and less than 1). The reasoning may be repeated for  $\mathcal{E} < 1$ , with the change of variables  $W = u\mathcal{E}^{1/2}$ . Let the minimum of  $\tilde{k}(\mathcal{E})$  over the whole range  $\mathcal{E} > 0$  be denoted by  $\tilde{k}_m$ . The  $\theta$ -time interval  $T$  contains  $r \geq q$  complete periods of the motion with energy  $\mathcal{E}$ , so that the time interval in which (A.8) is obeyed is at least  $r\tilde{k}_m T/(r+1)$ . Then, choosing:

$$k_T = \inf_{r \geq q} \frac{r\tilde{k}_m}{r+1} = \frac{q}{q+1} \tilde{k}_m$$

Statement A.1 is verified.

<sup>31</sup>Cf. Eq. (105); in this section we drop the index R.

<sup>32</sup> $T(\mathcal{E}) \rightarrow 0$  as  $\mathcal{E} \rightarrow \infty$ .



Now, consider the two equations (A.1) and:

$$\frac{d^2 W_f}{d\theta^2} + W_f + W_f^2 + \frac{W_f^3}{3} = 0 \quad (\text{A.10})$$

and motions  $W(\theta)$ , obeying (A.1), and  $W_f(\theta)$ , obeying (A.10), such that they have the same initial conditions at some “initial”  $\theta$ -time  $\theta_i > \theta_0$ . If we subtract (A.10) from (A.1), use the notations:

$$\begin{aligned} \rho &= W_f - W, \quad \sigma = \frac{dW_f}{d\theta} - \frac{dW}{d\theta}, \\ \eta &= |\rho| + |\sigma|, \end{aligned} \quad (\text{A.11})$$

and integrate the first order differential equations equivalent to (A.1) and (A.10) from  $\theta_i$  to  $\theta$ , we obtain the inequality<sup>33</sup>:

$$\begin{aligned} \eta &\leq \int_{\theta_i}^{\theta} \eta d\theta + \int_{\theta_i}^{\theta} \eta |W + W_f| d\theta \\ &\quad + \int_{\theta_i}^{\theta} \frac{\eta}{3} |W^2 + WW_f + W_f^2| d\theta \\ &\quad + \int_{\theta_i}^{\theta} \frac{3}{4\theta} \left| \frac{dW}{d\theta} \right| d\theta + \int_{\theta_i}^{\theta} |WG(\theta)| d\theta. \end{aligned} \quad (\text{A.12})$$

Because the energy  $\mathcal{E}$  of the “true” motion  $W(\theta)$  is bounded, all terms containing  $W, W_f$  explicitly are bounded by constants, so that we can represent (A.12) by:

$$\begin{aligned} \eta &< M \int_{\theta_i}^{\theta} \eta d\theta + M_1 \int_{\theta_i}^{\theta} \left( \frac{1}{\theta} + G(\theta) \right) d\theta \\ &\equiv M \int_{\theta_i}^{\theta} \eta d\theta + M_1 H(\theta) \end{aligned} \quad (\text{A.13})$$

with  $M, M_1$  suitable constants and  $H(\theta)$  monotonically increasing and positive,  $H(\theta_i) = 0$ . For  $\theta_i < \theta < \theta_i + T$ ,  $H(\theta) \leq \text{const} \times T/\theta_i$ . Gronwall’s inequality [Bellman, 1953, p. 35; Coddington & Levinson, 1955, p. 37] implies then, for  $\theta_i < \theta < \theta_i + T$ :

$$\eta(\theta) < M_1 H(\theta) \exp(MT) < \text{const} \frac{T}{\theta_i} \quad (\text{A.14})$$

where the constant is independent of  $\theta_i$ . We estimate now the energy loss  $\Delta\mathcal{E}$  of the “true” motion

$W(\theta)$  in a “ $\theta$ -time”  $T$ :

$$\begin{aligned} \Delta\mathcal{E} &= \int_{\theta_i}^{\theta_i+T} \frac{3}{4\theta} \left( \frac{dW}{d\theta} \right)^2 d\theta \\ &= \int_{\theta_i}^{\theta_i+T} \frac{3}{4\theta} \left( \frac{dW_f}{d\theta} \right)^2 d\theta \\ &\quad + \int_{\theta_i}^{\theta_i+T} \frac{3}{4} \left( \left( \frac{dW}{d\theta} \right)^2 - \left( \frac{dW_f}{d\theta} \right)^2 \right) d\theta \\ &> (k_T T) \frac{3}{4\theta_i} \mathcal{E}(\theta_i) - C\sqrt{\mathcal{E}} \int_{\theta_i}^{\theta_i+T} \eta(\theta) \frac{3}{4\theta} d\theta \\ &> k_T T \frac{3}{4(\theta_i + T)} \mathcal{E} - C_2 \frac{T^2}{\theta_i^2} \sqrt{\mathcal{E}}. \end{aligned} \quad (\text{A.15})$$

In the first step we used Statement A.1 and inequalities like (98) to bound  $|dW/d\theta|, |dW_f/d\theta|$  from above. Equation (A.14) was used in the second step. The total change of energy  $\mathcal{E}(\theta_i + T) - \mathcal{E}(\theta_i)$  is obtained by adding the increase due to the first term in (A.2). This latter is bounded by  $C_1 T \times \mathcal{E}(\theta_i)/\theta_i^3$ . One verifies that, if  $\theta_i$  is sufficiently large, the total change of energy is negative. We may even require that it be larger in absolute value than  $(3/4)k_1 \mathcal{E} T/\theta_i$ , for a number  $0 < k_1 < k_T$ . This gives a lower bound  $\mathcal{B}$  on the energies for which this may occur ( $\theta_i/(\theta_i + T) \approx 1$ ):

$$\sqrt{\mathcal{E}} > \sqrt{(\mathcal{B})} \equiv \frac{4}{3} \frac{C_2 T}{\theta_i} \frac{1}{k_T - k_1 - \frac{4C_1}{3\theta_i^2}}. \quad (\text{A.16})$$

This bound depends on  $\theta_i$  and decreases like  $1/\theta_i^2$ . We may assume that the maximal energy  $\mathcal{E}_m$  at  $\theta_i$  is such that (A.16) is satisfied and that even, say,<sup>34</sup>:

$$\mathcal{E}_m(\theta_i) > 2\mathcal{B}. \quad (\text{A.17})$$

Then at  $\theta = \theta_i + T$  the energies  $\mathcal{E}$  of all motions for which (A.16) is true at  $\theta_i$  have decreased at least to  $\mathcal{E}(1 - (3/4)k_1 T/\theta_i)$ , and so has the maximal

$$\mathcal{E}_m(\theta_i + T) = \mathcal{E}_m(\theta_i) \left( 1 - \frac{3}{4} \frac{k_1 T}{\theta_i} \right). \quad (\text{A.18})$$

This is a bound for the energies of all motions with  $\mathcal{E}(\theta_i) < \mathcal{E}_m(\theta_i)$ ; indeed, all those motions for which the inequality (A.16) at  $\theta_i$  is not obeyed cannot acquire by (A.5) in the  $\theta$ -time  $T$  sufficient energy to

<sup>33</sup> $\eta = 0$  at  $\theta_i$ .

<sup>34</sup>It is at our disposal to increase  $\mathcal{E}(\theta_i)$ , if necessary.

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get over  $\mathcal{E}_m(\theta_i + T)$ , in view of the condition (A.17) if  $\theta_i$  is large enough. Further, the decrease of the maximal energy  $\mathcal{E}_m(\theta)$  in the interval  $(\theta_i, \theta_i + T)$  is less than that of the bound  $\mathcal{B}$  in the same  $\theta$ -interval: at  $\theta_i + T$  the latter is, according to (A.16)

$$\mathcal{B}(\theta_i + T) \approx \mathcal{B}(\theta_i) \left( \frac{1 - 2T}{\theta_i} \right). \quad (\text{A.19})$$

Thus, (A.17) is obeyed also at  $\theta_i + T$  with the maximal energy (A.18) and the bound  $\mathcal{B}$  of (A.19); we may then proceed to  $\theta_i + 2T$ , etc. and conclude that, after  $n$  steps, the maximal energy is bounded by:

$$\mathcal{E}_m(\theta_i + nT) = \mathcal{E}_m(\theta_i) \prod_{j=1}^n \left( 1 - \frac{3}{4} \frac{k_1 T}{\theta_i + jT} \right). \quad (\text{A.20})$$

For large  $n$ , the product in (A.20) behaves like  $(nT/\theta_i)^{-3k_1/4} \approx (\theta/\theta_i)^{-3k_1/4}$ . We conclude that the maximal energy  $\mathcal{E}_m$  decreases like  $(\theta/\theta_i)^{-r}$ , with  $0 < r < 1$ . As a consequence, the weak bound on  $|w(\theta)|$  contained in the first inequality of (A.6) may be now strengthened to:

$$|w(\theta)| < \text{const} \left( \frac{\theta}{\theta_i} \right)^{3/8-r/2}. \quad (\text{A.21})$$

We return now to the energy  $E(\theta)$  of the original equation (95) and to the inequality (107):

$$\begin{aligned} \frac{dE}{d\theta} &< \frac{1}{3} \left| w(\theta)^3 \frac{dk}{d\theta} \right| \\ &< \text{const} \times \left( \frac{\theta}{\theta_i} \right)^{9/8-3r/2} \left( \frac{\theta_i}{\theta} \right)^{3/8} \frac{1}{\theta}. \end{aligned} \quad (\text{A.22})$$

Integration of (A.22) leads to:

$$E(\theta) - E(\theta_i) < \text{const} \times \frac{1}{\frac{3}{4} - \frac{3r}{2}} \left( \frac{\theta}{\theta_i} \right)^{3/4-3r/2} \quad (\text{A.23})$$

which justifies our assertion in Eq. (109) (with  $s = 3r/2$ ).

## B. The Inversion of the Averaging Transformations

A simple proof is offered that the averaging transformations (121a), (121b) (or (125a), (125b)) leading from  $(R, \phi)$  to  $(R_1, \phi_1)$  (or from the latter to  $(R_2, \phi_2)$ ) are invertible, if the quantity  $h(\theta_L)$ ,

Eqs. (91), (117) is small enough. This is achieved either for small enough  $\varepsilon$  or large enough  $|\theta_L|$ . In Sec. 6 the quantity  $h(\theta_L)$  is replaced in the averaging transformations by  $k(\theta_R)$ , which is monotonically decreasing like  $\theta_R^{-3/8}$  and may be thus made small for large  $\theta_R$ . We show that the transformation (121a), (121b) is one-to-one from a strip  $\{0 \leq R \leq M\}$  in the  $(R, \phi)$  plane to its domain of values in  $(R_1, \phi_1)$ , for small  $h$  (or  $k$ ). Here  $M$  is the bound on the values of  $R$  established in Sec. 4.7 (see Corollary 4.1).

From (121a), (121b), taking derivatives at fixed  $\theta_L$  one verifies that, for (small) positive constants  $C_{ij}, i, j = 1, 2$ , it is true that:

$$\left| \frac{\partial R_1}{\partial R} \right| \geq 1 - C_{11} h(\theta_L) M, \quad (\text{B.1a})$$

$$\left| \frac{\partial R_1}{\partial \phi} \right| \leq C_{12} h(\theta_L) M^2$$

$$\left| \frac{\partial \phi_1}{\partial R} \right| \leq C_{21} h(\theta_L), \quad (\text{B.1b})$$

$$\left| \frac{\partial \phi_1}{\partial \phi} \right| \geq 1 - C_{22} h(\theta_L) M.$$

Assume now two different points  $(R_a, \phi_a), (R_b, \phi_b)$  of the  $(R, \phi)$  plane were mapped to the same  $(R_1, \phi_1)$ . Consider then the two functions of  $s, 0 < s < 1$ :

$$\tilde{R}_1(s) \equiv R_1(R_a + sD \cos \alpha, \phi_a + sD \sin \alpha), \quad (\text{B.2})$$

$$\tilde{\phi}_1(s) \equiv \phi_1(R_a + sD \cos \alpha, \phi_a + sD \sin \alpha)$$

where  $D$  is the distance between the two points and  $\tan \alpha$  is the slope of the line joining them. Since the two functions  $\tilde{R}_1(s), \tilde{\phi}_1(s)$  assume the same value at  $s = 0$  and  $s = 1$ , there exist values  $0 < s_R, s_\phi < 1$  so that their derivatives with respect to  $s$  vanish there. Suppose  $\alpha$  is such that, e.g.  $|\cos \alpha| \geq 1/\sqrt{2}$ . Then, using (B.1a):

$$\begin{aligned} \left| \frac{d\tilde{R}_1}{ds}(s_R) \right| &> D \left| \frac{\partial R_1}{\partial R} \cos \alpha \right| - D \left| \frac{\partial R_1}{\partial \phi} \sin \alpha \right| \\ &> D \left( \frac{1}{\sqrt{2}} - C_{11} h M - C_{12} h M^2 \right). \end{aligned} \quad (\text{B.3})$$

It is clear that for  $h$  small enough, the right-hand side of (B.3) does not vanish, which contradicts the



fact that  $\tilde{R}_1(s)$  assumes the same value at  $s = 0$  and 1.

If  $|\cos \alpha| < 1/\sqrt{2}$ , we use the function  $\tilde{\phi}_1(s)$  of (B.2) and relation (B.1b) and reach the same conclusion.

In calculations, the inversion is achieved by expanding  $R(R_1, \phi_1)$ ,  $\phi(R_1, \phi_1)$  in powers of  $h$  (cf. Eqs. (123a), (123b)).

### C. The Solutions of the Variational Equation and the WKB Approximation

The following is an adaptation to the present situation of procedures that are common in the discussion of the WKB method (see, e.g. [Langer, 1949] and any classical book on differential equations, e.g. [Coddington & Levinson, 1955]).

#### C.1. The existence of some special solutions with WKB asymptotics

Consider the WKB functions (154):

$$V_{c,s}^{(as)}(\tau, \varepsilon) = \frac{3^{1/4}}{\Xi(\tau, \varepsilon)^{1/4}} \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} \left( \int_{\tau}^{\tau_a} \Xi(\tau, \varepsilon)^{1/2} d\tau \right) \quad (C.1)$$

for some  $\tau_a < 0$ . The functions  $w_{c,s}(\theta, \tau_a) \equiv \Xi^{1/4} V_{c,s}^{(as)}$  are the solutions  $\cos(\theta - \theta_a)$ ,  $\sin(\theta - \theta_a)$  of the equation

$$\frac{d^2 w}{d\theta^2} + w = 0, \quad \theta = - \int_{\tau}^{\tau_0} \Xi^{1/2}(\tau') d\tau' \quad (C.2)$$

and<sup>35</sup>  $\theta_a \equiv \theta(\tau_a)$ . The same changes of dependent and independent variable transform the variational equation (153) into

$$\frac{d^2 \tilde{w}}{d(\theta)^2} + \tilde{w}(1 + \tilde{G}(\theta)) = 0, \quad \tilde{w} \equiv \Xi^{1/4} V \quad (C.3)$$

with<sup>36</sup>

$$\tilde{G} = \frac{5}{16} \frac{\left( \frac{d\Xi}{d\tau} \right)^2}{\Xi^3} - \frac{d^2 \Xi}{d\tau^2} \Xi^2. \quad (C.4)$$

Solutions of (C.3) which assume at  $\tau = -\varepsilon^{-\delta}$  the values and derivatives of  $w_{c,s}$  are obtained using the method of variation of parameters as the unique solutions of the linear integral equation:  $(\theta(\varepsilon^{-\delta}) \approx -\varepsilon^{-4\delta/3})$

$$\tilde{w}_{c,s} = w_{c,s} + \int_{-\varepsilon}^{\theta} \tilde{G}(\theta') \sin(\theta - \theta') \tilde{w}_{c,s}(\theta') d\theta'. \quad (C.5)$$

Since  $\tilde{G}(\theta) \approx 1/\theta^2$ , the desired solution of (C.5) may be obtained by iteration at large  $|\theta|$  starting from  $w_{c,s}$ ; since its values and derivative are bounded for  $|\theta|$  sufficiently large (C.5), implies that, for such  $\theta$ ,

$$|(\tilde{w}_{c,s} - w_{c,s})(\theta)|, \quad \left| \frac{d\tilde{w}_{c,s}}{d\theta} - \frac{dw_{c,s}}{d\theta} \right| < \frac{C}{|\theta|}. \quad (C.6)$$

Reverting to the variable  $\tau$ , to the original  $V_{c,s}^{(as)}$  of (C.1) and to the sought solutions  $V_{c,s}(\tau, \varepsilon)$  of (153):

$$V_{c,s}(\tau(\theta), \varepsilon) \equiv \frac{\tilde{w}_{c,s}(\theta)}{\Xi(\tau(\theta), \varepsilon)^{1/4}} \quad (C.7)$$

we may state that, for all  $|\tau|$  sufficiently large (i.e. even larger than  $\varepsilon^{-\delta}$ ):

$$|(V_{c,s} - V_{c,s}^{(as)})(\tau)| < \frac{C}{|\tau|^{3/2}}, \quad (C.8)$$

$$\left| \left( \frac{dV_{c,s}}{d\tau} - \frac{dV_{c,s}^{(as)}}{d\tau} \right)(\tau) \right| < \frac{C}{|\tau|^{7/6}}$$

where we have used  $d\theta/d\tau \approx \tau^{1/3}$ . Equation (C.8) shows in what sense  $V_{c,s}(\tau)$  asymptotically approach  $V_{c,s}^{(as)}(\tau)$ . The solutions  $V_{c,s}$  obtained by (C.5) for large  $|\tau|$  may be extended down to  $\tau = 0$ .

#### C.2. The limit $\varepsilon \rightarrow 0$

The solutions  $V_{c,s}$  obtained above depend on the chosen value of  $\varepsilon$ . It is, however, plausible, as shown in Sec. 5.3 (cf. Eq. (156)) that they approach a limit as  $\varepsilon \rightarrow 0$ . Indeed, on an interval  $[-\varepsilon^{-\delta}, 0]$  with  $\delta = 3r/8$  (cf. Eq. (136)) the difference  $|\eta_L(\tau) - \eta_{00L}(\tau)|$  tends to zero as  $\varepsilon \rightarrow 0$ . The function  $\tilde{G}$ , Eq. (C.4) depends on  $\varepsilon$  through  $\eta_L(\tau)$  (cf. Eq. (155)), is well defined for  $\varepsilon = 0$  (replacing  $\eta_L$  by  $\eta_{00L}$ ) and

<sup>35</sup>The definition of  $\theta$  in (C.2) differs from the one of  $\theta_L$  in (84) by the implicit consideration of the term  $\gamma^2$  in the definition of  $\Xi$  in (155).

<sup>36</sup>The function  $\tilde{G}$  is equal to the first two terms in (92) if one uses the approximation (155) for  $\Xi(\tau, \varepsilon)$ .

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Eq. (C.5) has a formal limit for  $\varepsilon = 0$ :

$$\begin{aligned}\tilde{w}_{c,s}(\theta, 0) &= w_{c,s}(\theta) \\ &+ \int_{-\infty}^{\theta} \tilde{G}(\theta', 0) \sin(\theta - \theta') \tilde{w}_{c,s}(\theta', 0) d\theta'.\end{aligned}\quad (\text{C.9})$$

The solution  $\tilde{w}_{c,s}(\theta, 0)$  of (C.9) may be obtained by iteration and is bounded for large  $\theta$ ; we may estimate *at a fixed value of  $\theta$*  its departure from the solution  $\tilde{w}_{c,s}(\theta, \varepsilon)$  of (C.5) by Gronwall's Lemma as follows: we subtract (C.9) from (C.5), separate out the  $\tau$ -interval  $(-\infty, -\varepsilon^{-\delta})$  and estimate  $\Delta\tilde{w}_{c,s}(\theta) \equiv \tilde{w}_{c,s}(\theta, \varepsilon) - \tilde{w}_{c,s}(\theta, 0)$ :

$$\begin{aligned}|\Delta\tilde{w}_{c,s}(\theta)| &< \text{const} \times \int_{-\infty}^{\varepsilon^{-4/3}} \tilde{G}(\theta', 0) d\theta' + \text{const} \\ &\times \int_{\varepsilon^{-4/3}}^{\theta} |\tilde{G}(\theta', 0) - \tilde{G}(\theta', \varepsilon)| d\theta' \\ &+ \int_{-\varepsilon}^{\theta} \tilde{G}(\theta', \varepsilon) |\Delta\tilde{w}_{c,s}(\theta')| d\theta'.\end{aligned}\quad (\text{C.10})$$

Since  $\tilde{G} < C/\theta^2$ , the first term is bounded by  $\text{const} \times \varepsilon^{4\delta/3}$ . Using (C.4) and the expansions (49), (51), (cf. Eq. (74)) one verifies that:

$$|\tilde{G}(\theta, \varepsilon) - \tilde{G}(\theta, 0)| = O\left(\frac{\gamma}{\tau^{13/3}}, \frac{\varepsilon^{3/4}}{\tau^{2/3}}\right)$$

so that Eq. (C.10) may be rewritten in short:

$$|\Delta\tilde{w}_{c,s}(\theta)| < \tilde{C}(\varepsilon) + C \int_{-\varepsilon}^{\theta} \frac{1}{\theta'^2} |\Delta\tilde{w}(\theta')| d\theta' \quad (\text{C.11})$$

where  $\tilde{C}(\varepsilon)$  tends to zero when  $\varepsilon \rightarrow 0$ . Gronwall's Lemma [Bellman, 1953; Coddington & Levinson, 1955; Guckenheimer & Holmes, 1983], Ch. IV shows that

$$\begin{aligned}|\Delta\tilde{w}(\theta)| &< \tilde{C}(\varepsilon) \exp\left(\int_{-\varepsilon}^{\theta} \frac{C}{\theta'^2} d\theta'\right) \\ &< C_1(\varepsilon)\end{aligned}\quad (\text{C.12})$$

where  $C_1(\varepsilon)$  vanishes as  $\varepsilon$  tends to zero. This shows that, as announced, if the value of  $\theta$  is kept unchanged,  $w_{c,s}(\theta, \varepsilon)$  approach  $w_{c,s}(\theta, \varepsilon = 0)$  as  $\varepsilon \rightarrow 0$ . The same is true for the derivatives  $dw_{c,s}/d\theta(\theta, \varepsilon)$ . However, this does not yet imply that this limit exists *at fixed  $\tau$* : indeed the relation (C.2) defining  $\theta$  in terms of  $\tau$  depends through

$\Xi$ , Eq. (155), on  $\varepsilon$ , and if  $\tau$  is fixed,  $\theta(\tau, \varepsilon)$  is different from  $\theta(\tau, \varepsilon = 0)$ . We estimate the difference  $\Delta\theta(\tau, \varepsilon)$  between the  $\theta$ -values corresponding to the same  $\tau$  at  $\varepsilon$  and at  $\varepsilon = 0$ :

$$\begin{aligned}\Delta\theta &\equiv \theta(\tau, \varepsilon) - \theta(\tau, 0) \\ &= \int_{\tau_0}^{\tau} (\Xi(\tau', \varepsilon) - \Xi(\tau', 0)) d\tau' \\ &= O\left(\frac{\gamma}{\tau^{1/3}}, \varepsilon^{3/4} \tau^{10/3}\right)\end{aligned}\quad (\text{C.13})$$

where we have used the expansions (49), (51) to evaluate the integral. This difference vanishes as  $\varepsilon \rightarrow 0$  for  $|\tau| < \varepsilon^{-\delta}$  with the choice of  $\delta$  in (C.12). We can now write, using  $\theta_\varepsilon \equiv \theta(\tau, \varepsilon)$ ,  $\hat{\theta} \equiv \theta(\tau, 0)$

$$\begin{aligned}|\tilde{w}_{c,s}(\tau, \varepsilon) - \tilde{w}_{c,s}(\tau, 0)| &\leq |\tilde{w}_{c,s}(\theta_\varepsilon, \varepsilon) - \tilde{w}_{c,s}(\theta_\varepsilon, 0)| \\ &+ |\tilde{w}_{c,s}(\theta_\varepsilon, 0) - \tilde{w}_{c,s}(\hat{\theta}, 0)| \\ &\equiv T_1 + T_2.\end{aligned}\quad (\text{C.14})$$

The first term in (C.14) is the difference *at fixed  $\theta$*  and vanishes as  $\varepsilon \rightarrow 0$ , according to (C.12). For the second term we use the estimate (C.13) of  $\Delta\theta$  and the integral equation (C.9):

$$\begin{aligned}T_2 &\leq |w_{c,s}(\theta_\varepsilon) - w_{c,s}(\hat{\theta})| + \int_{-\infty}^{\theta_\varepsilon} \tilde{G}(\theta') (\sin(\theta_\varepsilon - \theta') \\ &- \sin(\hat{\theta} - \theta')) \tilde{w}_{c,s}(\theta', 0) d\theta' \\ &+ \int_{\hat{\theta}}^{\theta_\varepsilon} \tilde{G}(\theta') \sin(\hat{\theta} - \theta') \tilde{w}_{c,s}(\theta', 0) d\theta'.\end{aligned}\quad (\text{C.15})$$

All terms in (C.15) may be majorized by  $\text{const} \times |\theta_\varepsilon - \hat{\theta}|$  so that, as announced, at any fixed  $\tau$  in  $[\varepsilon^{-\delta}, 0]$ , the solutions of (C.5) approach those of (C.9) as  $\varepsilon \rightarrow 0$ .

### C.3. The difference of two solutions of the variational equation

We need sometimes an estimate of the difference of two solutions  $V_i(\tau)$ , ( $i = 1, 2$ ) of the variational equation on the whole interval  $[-\varepsilon^{-\delta}, 0]$ , knowing the difference of their initial values  $(\alpha_i, \beta_i)$ ,  $i = 1, 2$  at  $\tau = -\varepsilon^{-\delta}$ . Let  $\tilde{w}_i(\theta) \equiv V_i \Xi^{1/4}$  be then the solutions corresponding to two equations like (C.5), containing suitable initial conditions at  $\tau = -\varepsilon^{-\delta}$ .



These latter, which we call  $\hat{\alpha} \equiv w(-\varepsilon^{-\delta})$ ,  $\hat{\beta} \equiv dw/d\theta(-\varepsilon^{-\delta})$  are obtained from  $(\alpha_i, \beta_i)$  through:

$$\begin{aligned}\hat{\alpha}_i &= 3^{1/4} \alpha_i \varepsilon^{-\delta/6}, \\ \hat{\beta}_i &= 3^{-1/4} \varepsilon^{\delta/6} \beta_i + \frac{3^{-5/4}}{2} \varepsilon^{7\delta/6} \alpha_i.\end{aligned}\quad (\text{C.16})$$

Subtracting the two equations (C.5) corresponding to  $\tilde{w}_i$ ,  $i = 1, 2$  one obtains with an obvious notation:

$$\begin{aligned}\Delta \tilde{w}(\theta) &= \Delta \hat{\alpha} w_c(\theta) + \Delta \hat{\beta} w_s(\theta) \\ &+ \int_{-\varepsilon}^{\theta} \tilde{G}(\theta') \sin(\theta - \theta') \Delta \tilde{w}(\theta') d\theta'.\end{aligned}\quad (\text{C.17})$$

In a well known manner, from (C.17) and the boundedness of  $w_c, w_s$ , Gronwall's Lemma [Bellman, 1953; Coddington & Levinson, 1955] implies that:

$$|\Delta \tilde{w}(\theta)|, \quad \left| \Delta \frac{d\tilde{w}}{d\theta}(\theta) \right| = O(\max(\Delta \hat{\alpha}, \Delta \hat{\beta})). \quad (\text{C.18})$$

The estimate (C.18) holds as long as  $\Xi \neq 0$ , in our case down to  $\tau = 0$ . Reverting to the original  $V_i(\tau)$  and noticing that for finite  $\tau$ ,  $\Xi(\tau)$  is finite, together with its derivative, we conclude that the same estimate (C.18) is true also for the differences of  $V_i, dV_i/d\tau$ . Using (C.16):

$$|\Delta V(\tau)|, \quad \left| \Delta \frac{dV}{d\tau}(\tau) \right| = O(\max(\Delta \alpha \varepsilon^{-\delta/6}, \Delta \beta \varepsilon^{\delta/6})). \quad (\text{C.19})$$

#### D. The Limit of $R_{L,f}(\varepsilon)$ for $\varepsilon \rightarrow 0$

We show that the values  $R_{L,f}(\varepsilon) \equiv R(\varepsilon, t(\theta) = \pi/2)$  obtained through the solution of (176a) tend to a limit  $R_{L,f}(0)$  as  $\varepsilon \rightarrow 0$ . The latter is the asymptotic value of the solution obtained by setting formally  $\varepsilon = 0$  in (176a), (176b), i.e. with the replacements in  $k(\theta)$  indicated in Lemma 6.1:

$$\begin{aligned}\frac{dR_{L,0}}{d\theta} &= \frac{7}{216} \frac{R_{L,0}(\theta)}{\tau(\theta)^{8/3}} \sin(2z_0) \\ &+ \frac{R_{L,0}^2(\theta)}{4\tau(\theta)^{1/2}} (\sin z_0 + \sin 3z_0) \\ &+ \frac{R_{L,0}(\theta)^3}{12\tau(\theta)} \left( \sin 2z_0 + \frac{\sin 4z_0}{2} \right)\end{aligned}\quad (\text{D.1a})$$

$$\begin{aligned}\frac{d\phi_{L,0}}{d\theta} &= \frac{7}{216\tau(\theta)^{8/3}} (1 + \cos 2z) \\ &+ \frac{R_{L,0}(\theta)}{4\tau(\theta)^{1/2}} (3 \cos z_0 + \cos 3z_0) \\ &+ \frac{1}{12} \frac{R_{L,0}(\theta)^2}{\tau(\theta)} \left( \frac{3}{2} + 2 \cos 2z_0 + \frac{1}{2} \cos 4z_0 \right)\end{aligned}\quad (\text{D.1b})$$

where we have used the limiting forms (see Eqs. (92), (174))  $g(\theta) \approx (7/108)\tau(\theta)^{-8/3}$ ,  $k(\theta) \approx 1/\tau(\theta)^{1/2}$  and  $z_0 = \theta + \phi_{L,0}(\theta)$ . In (D.1a), (D.1b), one uses  $\tau(\theta)$  as determined from:

$$\theta(\tau) = \frac{3\sqrt{3}}{4} (\tau^{4/3} - \tau_0^{4/3}) \approx \frac{3\sqrt{3}}{4} \tau^{4/3} \quad (\text{D.2})$$

where the last approximation is true for large  $\tau$ . In this domain,

$$\tau(\theta) \approx \left( \frac{4\theta}{3\sqrt{3}} \right)^{3/4}. \quad (\text{D.3})$$

We perform now on Eqs. (D.1a), (D.1b) the same "averaging" transformations as in Sec. 5 and "eliminate" successively terms in  $1/\theta^{3/8}$  and  $1/\theta^{3/4}$ . At this latter stage, there appear in the equation for  $\phi_{L,0}$  secular terms, which may diverge as  $\theta \rightarrow \infty$ . It is convenient to continue this procedure and eliminate further terms in  $1/\theta^{9/8}$  at which stage it becomes apparent that further secular terms for  $\phi_{L,0}$  occur to  $O(1/\theta^{3/2})$ , but no such terms occur for  $R_{L,0}$ . A further transformation to remove terms in  $dk/d\theta \approx 1/\theta^{11/8}$  also leads in the equation for  $R_{L,0}$  to secular terms of  $O(kdk/d\theta) = O(1/\theta^{7/4})$ . None of these terms is divergent as  $\theta \rightarrow \infty$ . They contribute finite quantities to the phase  $\phi_{L,0}$ . If we denote by  $R_{4L,0}(\theta), \phi_{4L,0}(\theta)$  the dependent variables obtained after these four transformations, we find that these obey equations like:

$$\frac{dR_{4L,0}}{d\theta} = O(\theta^{-3/2}) \quad (\text{D.4a})$$

$$\begin{aligned}\frac{d\phi_{4L,0}}{d\theta} &= -\frac{7}{24} \left( \frac{3\sqrt{3}}{4} \right)^{3/4} \frac{R_{L,0}(\theta)^2}{\theta^{3/4}} \\ &+ O(\theta^{-3/2}).\end{aligned}\quad (\text{D.4b})$$

In writing these equations, we have used the fact that, according to Sec. 4,  $R_{L,0}$  is bounded for all  $\theta$  and that all other terms, coming from  $g(\theta)$  and the derivatives of  $k(\theta)$  fall off even quicker with  $\theta$ . Integrating (D.4a) between two values  $\theta_1$  and  $\theta_2$ , we

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conclude that:

$$|R_{4L,0}(\theta_2) - R_{4L,0}(\theta_1)| \leq \text{const}(\theta_1^{-1/2} - \theta_2^{-1/2}). \quad (\text{D.5})$$

Since for a sequence of points  $\theta_n \rightarrow \infty$  the difference between any two terms  $R_{4L,0}(\theta_n)$ ,  $R_{4L,0}(\theta_m)$  tends to zero when  $n, m \rightarrow \infty$ , it follows that the sequence  $R_{4L,0}(\theta_n)$  approaches a limit, which we call  $R_{L,f}(0)$ . We return now step by step to the original variable  $R_{L,0}(\theta)$  by inverting the transformations leading to  $R_{4L,0}$ ; these transformations are such that the difference between  $R_{iL,0}$  and  $R_{jL,0}$  decreases to zero as  $\theta \rightarrow \infty$  so that we conclude that  $R_{L,0}(\theta)$  approaches the same limit  $R_{L,f}(0)$ .

Further, for a finite value of  $\varepsilon$ , for which  $\theta(t = \pi/2) = \text{const}/\sqrt{\varepsilon}$ , we perform the same transformations to obtain equations similar to (D.4a) and (D.4b), with the difference that on the right-hand side, we have powers of  $k(\theta)$ , and its product with derivatives or with  $g(\theta)$ :

$$\frac{dR_{4L,\varepsilon}}{d\theta} = O(k^4 + g) \quad (\text{D.6a})$$

$$\frac{d\phi_{4L,\varepsilon}}{d\theta} = -\frac{7}{24}R_{L,\varepsilon}(\theta)^2k(\theta)^2 + O(k^4 + g). \quad (\text{D.6b})$$

It is necessary to keep the terms with  $g(\theta)$  because  $k(\theta)$  falls off exponentially for time scales of  $O(1)$  and thus may become smaller than  $g(\theta)$ , which settles to a value of  $O(\varepsilon(\ln(1/\varepsilon))^2)$  after falling off like  $O(\theta^{-2})$  when  $t = o(1)$ . Integrating (D.6a) between a sufficiently large value of  $\theta$  and  $\theta_f \equiv \theta(t = \pi/2)$  we obtain:

$$|R_{4L,\varepsilon}(\theta_f) - R_{4L,\varepsilon}(\theta)| \leq \text{const} \int_{\theta}^{\theta_f} (k(\theta)^4 + g(\theta))d\theta. \quad (\text{D.7})$$

If we choose  $|\tau| > \varepsilon^{-\delta}$  for  $0 < \delta < 3/8$ , the integrals in Eq. (D.7) are less than  $\text{const} \times \varepsilon^{2\delta/3}$ ; if  $\delta = 3/8$  (i.e.  $t = O(1)$ ), they are even less than  $\text{const}\sqrt{\varepsilon}(\ln(1/\varepsilon))^2$  (the terms with  $g(\theta)$  are dominant).

Thus, we must now show that the difference of the solutions of the limiting equations (D.4a) and (D.4b) to those of the exact equations (D.6a) and (D.6b) at such values of  $\theta$  vanishes as  $\varepsilon \rightarrow 0$ . To this end, we write (D.4a), (D.4b), (D.6a), (D.6b) in a more explicit form (the precise values of the

coefficients may be obtained from an algebraic manipulation program, but they are of no importance for the present purpose). For instance, (D.6a) reads:

$$\frac{dR_{4L,\varepsilon}}{d\theta} = k(\theta)^4 R_{L,\varepsilon}(\theta)^5 P_1(z) + \dots \quad (\text{D.8})$$

where  $P_1(z)$  is a trigonometric polynomial of  $z = \theta + \phi_{L,\varepsilon}(\theta)$ . The further terms contain higher powers of  $k(\theta)$  and products of  $k(\theta)$  or its derivatives with  $g(\theta)$ . Equation (D.4a) is similar: the same polynomials in  $z$  occur, but (as in (D.1a), (D.1b)),  $X_R(t)$  is replaced by the leading term of the inner expansion (63), i.e. for large  $\theta$ ,  $k(\theta) \approx \text{const}/\theta^{3/8}$ . From Eqs. (49), (65) we see that, for large  $\tau$ , the difference:

$$\eta_R(\tau) - \tau^{1/3} = O\left(\varepsilon^{3/4}\tau^{7/3}, \frac{\gamma}{\tau^{4/3}}\right). \quad (\text{D.9})$$

Equation (D.9) allows us to estimate the differences between the various coefficients of Eq. (D.8) and its analogon (D.4a). We notice that the differences have to be estimated at a fixed value of  $\theta$ , which corresponds to different values of  $\tau$  denoted by  $\tau(0, \theta)$ ,  $\tau(\varepsilon, \theta)$  in turn, in the situations  $\varepsilon = 0$  and finite  $\varepsilon$  (cf. also Appendix C, Eq. (C.13)). The value of  $\tau$  corresponding to  $\theta$  at  $\varepsilon = 0$  is given by (D.3) but for finite  $\varepsilon$ , it is the solution of

$$\begin{aligned} \theta &\cong \sqrt{3} \int_{\tau_0}^{\tau} \eta_R(\tau') d\tau' \\ &\cong \frac{3\sqrt{3}}{4} \tau^{4/3} (1 + O(\varepsilon^{3/4}\tau^2)) \end{aligned} \quad (\text{D.10})$$

where we have used the estimate (D.9) for  $\eta_R(\tau)$ . For small  $\varepsilon$  and  $\tau < \varepsilon^{-\delta}$ ,  $\delta < 3/8$ , one obtains:

$$\tau(\varepsilon, \theta) = \tau(0, \theta) (1 + O(\varepsilon^{3/4}\tau(0, \theta)^2)). \quad (\text{D.11})$$

Using this in (D.9) one gets an estimate:

$$\eta_R(\tau(\varepsilon, \theta)) = \tau(0, \theta)^{1/3} (1 + O(\varepsilon^{3/4}\tau(0, \theta)^2)) \quad (\text{D.12})$$

which shows that in evaluating orders of magnitudes of differences at fixed  $\theta$ , one may replace  $\tau(\varepsilon, \theta)$  by  $\tau(0, \theta)$  of (D.3). Let

$$\begin{aligned} \Delta R_{4L}(\theta) &\equiv R_{4L,\varepsilon}(\theta) - R_{4L,0}(\theta) \\ \Delta R_L(\theta) &\equiv R_{L,\varepsilon}(\theta) - R_{L,0}(\theta) \end{aligned} \quad (\text{D.13a})$$

$$\bar{\phi}_{4L,\varepsilon}(\theta) \equiv \phi_{4L,\varepsilon}(\theta) + \frac{7}{24} \int_0^{\theta} k(\theta)^2 R_L(\theta)^2 d\theta \quad (\text{D.13b})$$



$$\begin{aligned}\Delta\phi_{4L}(\theta) &\equiv \phi_{4L,\varepsilon}(\theta) - \phi_{4L,0}(\theta) \\ \Delta\phi_L(\theta) &\equiv \phi_{L,\varepsilon}(\theta) - \phi_{L,0}(\theta)\end{aligned}\quad (\text{D.13c})$$

and  $\bar{\phi}_{L,\varepsilon}$  be related to  $\phi_{L,\varepsilon}$  by (179). Subtracting from each other equations like (D.8) and their counterparts for  $\varepsilon = 0$  and using the uniform boundedness with respect to  $\varepsilon$  of  $R_L(\theta)$  one obtains

$$\begin{aligned}\frac{d\Delta R_{4L}}{d\theta} &\leq \text{const } \Delta(k(\theta)^4) \\ &\quad + a_1(\theta)\Delta R_L + b_1(\theta)\Delta\phi_L\end{aligned}\quad (\text{D.14a})$$

$$\begin{aligned}\frac{d\Delta\bar{\phi}_{4L}}{d\theta} &\leq \text{const } \Delta(k(\theta)^4) \\ &\quad + a_2(\theta)\Delta R_L + b_2(\theta)\Delta\phi_L\end{aligned}\quad (\text{D.14b})$$

$$\begin{aligned}\Delta\phi_{4L} &\leq \Delta\bar{\phi}_{4L} \\ &\quad + \frac{7}{24}\Delta\int_0^\theta k(\theta')^2 R_L(\theta')^2 d\theta'.\end{aligned}\quad (\text{D.14c})$$

In Eqs. (D.14a), (D.14b), the functions  $a_i(\theta), b_i(\theta)$  are sums over the various  $\theta$ -dependent coefficients in (D.6a), (D.6b) multiplied by constants obtained from the simple estimates  $|P_i(z)| < \text{const}$ . As long as  $t$  is confined to an interval  $(\varepsilon^{3/8} < t < C_\varepsilon), C_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , the dominant term is of  $O(k^4) = O(\theta^{-3/2})$ , so that we may assume this is the order of magnitude of  $a_i(\theta), b_i(\theta)$  in (D.14a), (D.14b). From a repeated application of the “inverse” equations (123a), (123b) we may express  $\Delta R_L, \Delta\phi_L$  in terms of  $\Delta R_{4L}, \Delta\phi_{4L}$  through:

$$\begin{aligned}\Delta R_L &= \Delta R_{4L}(1 + O(k(\theta))) \\ &\quad + \Delta\phi_{4L}O(k(\theta)) + O(\Delta k(\theta))\end{aligned}\quad (\text{D.15a})$$

$$\begin{aligned}\Delta\phi_L &= \Delta R_{4L}O(k(\theta)) + \Delta\phi_{4L}(1 + O(k(\theta))) \\ &\quad + O(\Delta k(\theta)).\end{aligned}\quad (\text{D.15b})$$

Upon substitution in (D.14a), (D.14b), one verifies that the estimates remain unchanged if we replace  $\Delta R_L, \Delta\phi_L$  by  $\Delta R_{4L}, \Delta\phi_{4L}$  and also the order of magnitude of the coefficients is preserved (we keep the notation unchanged). To replace further  $\Delta\phi_{4L}$  by  $\Delta\bar{\phi}_{4L}$  we need a bound on the integral in (D.14c). If  $M$  is a bound on  $R_L, R_{L0}$  we obtain, using (D.15a), (D.15b):

$$\begin{aligned}\int_0^\theta \Delta(k^2 R_L)^2 d\theta' \\ \leq M^2 \int_0^\theta (\Delta k)^2 d(\theta') + 2M \int_0^\theta k^2 \Delta R_L(\theta') d\theta'\end{aligned}$$

$$\begin{aligned}&\leq M^2 \int_0^\theta (\Delta k)^2 d(\theta') + C_1 \int_0^\theta k^2 \Delta R_{4L} d\theta' \\ &\quad + C_2 \int_0^\theta k^3 \Delta\bar{\phi}_{4L} d\theta' \\ &\quad + C_3 \int_0^\theta k(\theta')^3 d\theta' \int_0^{\theta'} \Delta(k^2 R_L^2) d\theta''\end{aligned}\quad (\text{D.16})$$

with  $C_1, C_2, C_3$  constants pertaining to the  $O()$  terms in (D.15a), (D.15b). If we invert the order of integration in the last term and realize that, for a sufficiently high  $\tau_0$ , it is true that, for all  $0 < \theta' < \theta$ :

$$\begin{aligned}1 - C_3 \int_{\theta^0}^\theta k(\theta'')^3 d\theta'' \\ \geq 1 - C_3 \int_0^\theta k^3 d\theta'' \geq 1 - \frac{C_3}{\tau_0^{1/6}} \\ > \text{const} > 0\end{aligned}\quad (\text{D.17})$$

we may conclude that:

$$\begin{aligned}\int_0^\theta \Delta(k^2 R_L^2) d\theta' \\ \leq \text{const} \left( \int_0^\theta (\Delta k)^2 d\theta' + \int_0^\theta k^2 \Delta R_{4L} d\theta' \right. \\ \left. + \int_0^\theta k^3 \Delta\bar{\phi}_{4L} d\theta' \right).\end{aligned}\quad (\text{D.18})$$

This allows us to obtain inequalities like (D.14a), (D.14b) only in terms of  $R_{4L}, \bar{\phi}_{4L}$ . We integrate now the resulting inequality in (D.14a) from  $\theta = 0$  to  $\theta$  and interchange the order of integration. One obtains:

$$\begin{aligned}|\Delta R_{4L}(\theta)| &\leq |\Delta R_{4L}(0)| + T(\theta) \\ &\quad + \int_0^\theta c_1(\theta, \theta') \Delta R_{4L}(\theta') d\theta' \\ &\quad + \int_0^\theta c_2(\theta, \theta') \Delta\bar{\phi}_{4L}(\theta') d\theta'\end{aligned}\quad (\text{D.19})$$

where the dominant terms are now originating in (D.16): e.g. using the notation in (D.14a)

$$\begin{aligned}T(\theta) &= \int_0^\theta \Delta(k^2)(\theta') d\theta' \int_{\theta^0}^\theta b_1(\theta'') d\theta'' \\ &= O(\varepsilon^{3/4} \theta^{5/4})\end{aligned}\quad (\text{D.20})$$

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and

$$\begin{aligned} |c_1(\theta, \theta')| &< k(\theta')^2 \int_{\theta^0}^{\theta} a_1(\theta'') d\theta'' \\ &= O(\theta'^{-5/4}). \end{aligned} \quad (\text{D.21})$$

In (D.20) we have used the estimate (D.12) of  $\eta_R(\tau)$ :

$$\begin{aligned} \Delta k(\theta)^2 &= k(\theta)^2 - \frac{1}{(\tau(\theta)^{1/2})^2} \\ &= \text{const} \frac{1}{\tau(\theta)} \left( \frac{1}{(1 + \varepsilon^{3/4} \tau^2)^3} - 1 \right) \\ &= O(\varepsilon^{3/4} \tau). \end{aligned} \quad (\text{D.22})$$

Estimates similar to (D.21) hold for  $c_2(\theta, \theta')$  and for the coefficients appearing in the inequality for  $\Delta \bar{\phi}_{4L}$  analogous to (D.19). Defining:

$$\Delta \sigma(\theta) \equiv |\Delta R_{4L}(\theta)| + M |\Delta \bar{\phi}_{4L}(\theta)| \quad (\text{D.23})$$

and adding (D.19) and its analogon for  $\Delta \bar{\phi}_{4L}$  one obtains:

$$\Delta \sigma(\theta) \leq T_1(\theta) + \int_0^{\theta} d(\theta') \Delta \sigma(\theta') d\theta' \quad (\text{D.24})$$

where  $T_1(\theta)$  obeys an estimate like (D.20) and  $d(\theta)$  an estimate like (D.21). Gronwall's inequality implies then:

$$\Delta \sigma(\theta) = O(\varepsilon^{3/4} \theta^{5/4}) \quad (\text{D.25})$$

and thus vanishes as  $\varepsilon \rightarrow 0$  on intervals of  $\theta$  of order  $\varepsilon^{-4\delta/3}$  with  $\delta < 3/8$ . With this, the difference between the value  $R_{4L,\varepsilon}(t = \pi/2) \equiv R_{4L,f}(\varepsilon)$  and the asymptotic value  $R_{4L,f}(0)$  obtained from the equation with  $\varepsilon = 0$  is:

$$\begin{aligned} |R_{4L,f}(\varepsilon) - R_{4L,f}(0)| \\ \leq |R_{4L,f}(\varepsilon) - R_{4L,\varepsilon}(\theta)| \\ + |R_{4L,\varepsilon}(\theta) - R_{4L,0}(\theta)| \\ + |R_{4L,0}(\theta) - R_{4,f}(0)|. \end{aligned} \quad (\text{D.26})$$

For  $\theta = O(\varepsilon^{-4\delta/3})$ ,  $\delta < 3/8$ , the first term is  $O(\varepsilon^{2\delta/3})$  (cf. the comments on (D.7)), the second term is  $O(\varepsilon^{3/4-5\delta/3})$  (cf. Eq. (D.25)) and the third term is also of  $O(\theta^{-1/2}) = O(\varepsilon^{2\delta/3})$ . For any choice of  $\delta < 3/8$ , the difference (D.26) vanishes as  $\varepsilon \rightarrow 0$ . Inverting the transformations leading from  $R_L(\theta)$  to  $R_{4L}(\theta)$ , this result holds also for the original variables  $R_L, R_{L,0}$ , because  $k(\theta)$  approaches  $k(\varepsilon = 0, \theta) \equiv 1/\tau(\theta)^{1/2}$  as  $\varepsilon \rightarrow 0$  on the whole  $\tau$ -interval  $[1, \pi/2/\varepsilon^{3/8}]$ . We obtain thus the statement of Lemma 6.1. From this argument it follows

that the asymptotic value of  $R_L(\theta)$  is approached closely even in the "transition domain" between the boundary layer region and where  $t = O(1)$ . This limiting value for  $R_L$  may be obtained once for all by solving the boundary layer equation for  $X_L(t)$  for  $t > 0$  using for  $X_R(t)$  the simple approximation  $X_R(t) \approx \varepsilon^{1/8} \eta_{00}(\tau)$  and looking at the asymptotic value of  $R_L(\theta)$ . It turns out to be  $\approx 0.844$ . Clearly, the same argument as above serves to show that the phase difference  $\Delta \bar{\phi}_{4L}(\theta)$  tends to zero as  $\varepsilon \rightarrow 0$  and thus justifies the contents of Lemma 6.3. The limiting value  $\bar{\phi}_{L,f}(0)$  depends on the choice of  $\tau_0$  (the origin of the variable  $\theta_R$ ; for  $\tau_0 = 15$ ,  $\bar{\phi}_{L,f}(0) \approx 0.14$ ).

## E. The Evaluation of Some Integrals

In this Appendix we drop for simplicity the index  $L$  used in Sec. 6:  $R(\theta) \equiv R_{L,\varepsilon}(\theta)$ , etc. To evaluate (217) we add and subtract  $R(\theta(\pi/2))(\partial R/\partial R_0)(\theta)$  and we are thus led to the evaluation of the integrals:

$$\mathcal{I}_1 \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 \frac{\partial R}{\partial R_0}(\theta) \left( R(\theta) - R\left(\theta\left(\frac{\pi}{2}\right)\right) \right) d\theta \quad (\text{E.1})$$

$$\begin{aligned} \mathcal{I}_2 \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 R(\theta) \\ \times \left[ \left( \frac{\partial R}{\partial R_0} \right)(\theta) - \left( \frac{\partial R}{\partial R_0} \right)\left(\theta\left(\frac{\pi}{2}\right)\right) \right] d\theta. \end{aligned} \quad (\text{E.2})$$

For  $\mathcal{I}_1$  we use the boundedness of  $\partial R/\partial R_0$  and the relation:

$$R(\theta) = R_4(\theta) + O(k(\theta)) \quad (\text{E.3})$$

obtained by iteration of (123a), (123b) to obtain:

$$\begin{aligned} \mathcal{I}_1 &< \text{const} \int_0^{\theta(\pi/2)} k(\theta)^2 \left( \left| R_4(\theta) - R_4\left(\theta\left(\frac{\pi}{2}\right)\right) \right| \right. \\ &\quad \left. + O(k(\theta)) \right) d\theta \\ &< \text{const} \left( \int_0^{\theta(\pi/2)} k(\theta)^3 d\theta \right) \\ &\quad + O \left( \int_0^{\theta(\pi/2)} k(\theta')^2 d\theta' \int_{\theta^0}^{\theta(\pi/2)} k(\theta'')^4 d\theta'' \right) \end{aligned} \quad (\text{E.4})$$



where we have used (D.7) to evaluate the difference  $|R_4(\theta) - R_4(\theta(\pi/2))|$  and the fact that  $\delta R(\theta)$  is bounded (see text following Eq. (198)). Now, both integrals in (E.4) are bounded so that we deduce:

$$\mathcal{I}_1 < \infty. \quad (\text{E.5})$$

Using Lemma 6.4 we may also conclude that  $\mathcal{I}_1$  has a limit as  $\varepsilon \rightarrow 0$ .

For  $\mathcal{I}_2$  we write, using (191a):

$$\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22} \quad (\text{E.6})$$

$$\begin{aligned} \mathcal{I}_{21} = \int_0^{\theta(\pi/2)} k(\theta)^2 R(\theta) & \left[ \left( \frac{\partial R_4}{\partial R_0} \right) (\theta) (1 + O(k)) \right. \\ & \left. - \left( \frac{\partial R_4}{\partial R_0} \right) \left( \theta \left( \frac{\pi}{2} \right) \right) \left( 1 + O \left( k \left( \frac{\pi}{2} \right) \right) \right) \right] d\theta \end{aligned} \quad (\text{E.7})$$

$$\begin{aligned} \mathcal{I}_{22} & \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 R(\theta) \left[ \left( \frac{\partial \phi_4}{\partial R_0} (\theta) \right) O(k(\theta)) \right. \\ & \left. - \left( \frac{\partial \phi_4}{\partial R_0} \left( \theta \left( \frac{\pi}{2} \right) \right) \right) O \left( k \left( \theta \left( \frac{\pi}{2} \right) \right) \right) \right] d\theta \\ & \equiv \mathcal{I}_{221} + \mathcal{I}_{222}. \end{aligned} \quad (\text{E.8})$$

In (E.7) we replace the difference of the values of  $\partial R_4 / \partial R_0$  at  $\theta$  and  $\theta(\pi/2)$  by the integral over the right-hand side of (190a). Since both  $R(\theta)$  and  $\partial R / \partial R_0(\theta)$  are bounded, the integral over the first term in (190a) converges. The second term is more difficult, since  $\partial \phi / \partial R_0$  diverges like  $\theta^{1/4}$ , so that the integral from  $\theta$  to  $\infty$  falls off like  $1/\theta^{1/4}$ ; this brings a contribution behaving like  $\ln(1/\gamma)$  in  $\mathcal{I}_{21}$ . To improve on this, we use below the fact that the polynomial  $\bar{P}_2(z)$  has zero mean. We expect namely that the oscillations of  $\bar{P}_2(z)$  reduce the magnitude of the integrals. Before proceeding, we move to  $\mathcal{I}_{22}$ , where a similar problem occurs. The second term of the integral in (E.8) is negligible, since  $k(\theta(\pi/2))$  is exponentially small. In the first term, the factor  $\partial \phi_4 / \partial R_0$  increases like  $\theta^{1/4}$  (see Eq. (204)) so that the power of  $1/\theta$  under the integral is at first sight  $7/8$ , which is not enough for convergence. However, as remarked in relation to Eqs. (191a), (191b), the true appearance of this term is (we leave out the last term):

$$\mathcal{I}_{221} = \int_0^{\theta(\pi/2)} k(\theta)^3 R(\theta)^2 \frac{\partial \phi_4}{\partial R_0} (\theta) T(z_4) d\theta \quad (\text{E.9a})$$

$$z_4 = \theta + \phi_4(\theta) \quad (\text{E.9b})$$

where  $T(z_4)$  is a trigonometric polynomial with zero mean (in this case,  $T(z_4) = \sin z_4 + (1/3) \sin(3z_4)$ , cf. Eq. (123a), (123b). The property of zero mean ensures that there exists another trigonometric polynomial (which may be also chosen to have zero mean)  $S(z_4)$  so that:

$$\frac{dS}{dz_4} = T(z_4). \quad (\text{E.10})$$

We transform then  $\mathcal{I}_{221}$  by partial integration:

$$\begin{aligned} \mathcal{I}_{221} & = k(\theta)^3 R(\theta)^2 \frac{\partial \phi_4}{\partial R_0} \frac{1}{1 + \frac{d\phi_4}{d\theta}} S(z_4) \Bigg|_0^{\theta(\pi/2)} \\ & - \int_0^{\theta(\pi/2)} \frac{d}{d\theta} \left[ k^3(\theta) R^2(\theta) \frac{\partial \phi_4}{\partial R_0} \frac{1}{1 + \frac{d\phi_4}{d\theta}} \right] \\ & \times S(z_4) d\theta. \end{aligned} \quad (\text{E.11})$$

Now, the derivatives with respect to  $\theta$  in Eq. (E.11) are affected by the increase of the rate of falloff of the integrand with respect to  $\theta$ : indeed,

$$\frac{dR}{d\theta} = O(k), \quad \frac{d}{d\theta} \frac{\partial \phi_4}{\partial R_0} = O(k^2)$$

as follows from Eqs. (176a) and (190b), and

$$\frac{d^2 \phi_4}{d\theta^2} = O(k^2)$$

as follows from Eq. (D.6b). Further, the denominator  $1 + d\phi_4/d\theta$  is nonvanishing if only we take the starting point  $\tau_0$  sufficiently large. As a consequence, the integrals containing each of these derivatives are absolutely convergent. The first term is clearly bounded, so that we conclude that  $\mathcal{I}_{22}$  is itself bounded. The same argument may be applied to the second term in  $\mathcal{I}_{21}$ : a partial integration increases the falloff rate of the integral from  $\theta$  to  $\infty$  and ensures thus the boundedness of  $\mathcal{I}_{21}$ . Moreover, after performing the partial integration, it is possible to take the limit  $\varepsilon \rightarrow 0$ . Indeed all integrals can be seen to make sense and be finite even if we set  $\varepsilon = 0$  formally and replace all quantities with superscript 0 (corresponding to  $\varepsilon = 0$ , see Sec. 6.2). Moreover, the values of the integrals tend to those for  $\varepsilon = 0$  as  $\varepsilon \rightarrow 0$ . To see this, one notices that: (i) since all integrals extending from 0 to  $\infty$  are absolutely convergent, the contribution of the interval  $(\varepsilon^{-s}, \theta(\pi/2))$ ,  $s > 0$  may be made arbitrarily small,

for  $\varepsilon$  small enough. (ii) The departure of the various quantities of interest ( $R(\theta)$ ,  $\partial R/\partial R_0(\theta)$ , etc.) from their values for  $\varepsilon = 0$  (indexed with a superscript “0” in Sec. 6.2) is estimated by expressions like  $\varepsilon^a \theta^b$  with  $a > 0, b \geq 0$  (see Eqs. (211), (D.25)). Their integrals over intervals  $(0, \varepsilon^{-s})$  are quantities of  $O(\varepsilon^a \varepsilon^{-(b+1)s})$ . This is a positive power of  $\varepsilon$  if  $s < a/(b+1)$ . This shows that, indeed, all integrals are continuous at  $\varepsilon = 0$ .

## F. On the Invariant Sets of the Circle Map $\Pi$

In this Appendix, a proof is given for Lemma 7.2.

### F.1. General comments

- (i) We restrict ourselves to  $0 < \beta < \pi$ ; it follows that we can assume  $-\pi < \chi < \pi$ . It is of some advantage to assume  $-\pi/2 < \Sigma < 3\pi/2$  and to write:

$$\Sigma = \frac{\pi}{2} + \hat{\Sigma}, \quad \Pi(\chi) = -\beta \sin(\chi + \hat{\Sigma}). \quad (\text{F.1})$$

With this, Eqs. (248)–(250) for  $\beta_{2u}$ ,  $\beta_{2d}$  are changed to the unified form:

$$\begin{aligned} \beta_{2u} \sin(\beta_{2u} + \hat{\Sigma}) &= \frac{\pi}{2} + \hat{\Sigma} \pmod{(2\pi)}, \\ \beta_{2u} \neq \frac{\pi}{2} - \hat{\Sigma} &\equiv \beta_{1u} \pmod{(2\pi)} \end{aligned} \quad (\text{F.2})$$

$$\begin{aligned} \beta_{2d} \sin(\beta_{2d} - \hat{\Sigma}) &= \frac{\pi}{2} - \hat{\Sigma} \pmod{(2\pi)}, \\ \beta_{2d} \neq \frac{\pi}{2} - \hat{\Sigma} &\equiv \beta_{1d} \pmod{(2\pi)}. \end{aligned} \quad (\text{F.3})$$

The solutions  $\beta_{2u}$ ,  $\beta_{2d}$  correspond to (“super-stable”) period two orbits passing through the maxima (at  $\chi_M = -\pi/2 - \hat{\Sigma}$ ) and minima (at  $\chi_m = \pi/2 - \hat{\Sigma}$ ) of  $\Pi(\chi)$ , Eq. (F.1).<sup>37</sup> The specification  $\pmod{2\pi}$  means for  $\beta_{2u}$ ,  $\beta_{2d}$  that the quantity must be transferred to the interval  $(-\pi, \pi)$  through suitable addition or subtraction of  $2\pi$ . The values of  $\beta_{1u}$ ,  $\beta_{1d}$  are also solutions of the equations satisfied by  $\beta_{2u}$ ,  $\beta_{2d}$  and immediately precede the latter. They play a role only if they are positive and less than  $\pi$

(after  $2\pi$  translation). Equations (F.2), (F.3) show the symmetry:

$$\beta_{2u}(-\hat{\Sigma}) = \beta_{2d}(\hat{\Sigma}). \quad (\text{F.4})$$

In Fig. 14 we show the appearance of the solutions  $\beta_{2u,2d}(\Sigma)$  in the  $\Sigma - (-\beta)$  plane of Fig. 11. The right-hand branch (with respect to  $\hat{\Sigma} = 0$ ) corresponds to  $\beta_{2d}(\Sigma)$ , the left-hand one (small squares) to  $\beta_{2u}(\Sigma)$ .

- (ii) For any  $\alpha \in (-\beta, \beta)$ , the equation  $\Pi(\beta, \hat{\Sigma}; \chi) = \alpha$  has two solutions with  $-\pi < \chi < \pi$ . If  $\alpha = \chi_+(\chi_-)$  is a fixed point of  $\Pi$ , we denote its pair by  $\tilde{\chi}_+(\tilde{\chi}_-)$  (if the fixed point is positive, it is denoted by  $\chi_+ > 0$ , otherwise by  $\chi_- < 0$ ).
- (iii) According to (236),  $S(\Pi) < 0$ . This has the consequences (see [Collet & Eckmann, 1983, p. 97]): (a)  $S(\Pi^p) < 0$ <sup>38</sup> for all  $p > 0$ ; (b)  $|d(\Pi^p)/d\chi|$  cannot have a strictly positive minimum; (c) if  $d\Pi^p/d\chi$  does not change sign for  $\chi \in [a, b]$  and  $\Pi^p$  has three fixed points there, the middle one is unstable and the other two are stable; (d)  $\Pi^p$  cannot have more than three fixed points in an interval  $[a, b]$  where  $d\Pi^p/d\chi > 0(< 0)$ .

### F.2. The situation $\hat{\Sigma} =$

At  $\hat{\Sigma} = -\pi$  ( $\Sigma = -\pi/2$ ),  $\Pi$  may be decomposed into two maps  $\Pi_+$ ,  $\Pi_-$  of  $[-\beta, 0]$ ,  $[0, \beta]$  into themselves. From (F.2), (F.3) one verifies that  $\beta_{1u} = \beta_{1d} = \pi/2$ ,  $\pi/2 < \beta_{2u}, \beta_{2d} < \pi$ .

If  $\beta < 1$ , only  $\chi = 0$  is a fixed point and  $\Pi$  is a contraction (Lemma 7.1). If  $\beta > 1$ , there are three fixed points,  $\chi_-$ ,  $\chi_0 = 0$ ,  $\chi_+$ ;  $\chi_0$  is unstable. If  $\beta \leq \pi/2$ ,  $\Pi_+$  contracts the interval  $[0, \chi_+]$  into itself and to  $\chi_+$ . Indeed, on  $[0, \chi_+]$ , (i)  $\Pi_+(\chi)$  is monotonically increasing, (ii)  $\Pi_+(\chi) - \chi > 0$  and vanishes only at the ends, (iii)  $\Pi_+(\chi) < \chi_+$  since  $0 = \Pi_+(\chi_+) - \chi_+ > \Pi_+(\chi) - \chi_+$ . It follows that, for any  $\chi \in [0, \chi_+]$  the monotonically increasing sequence  $\{\chi, \Pi(\chi), \Pi^2(\chi) \dots\}$  has a limit which can only be  $\chi_+$ . Further, the interval  $[\chi_+, \beta]$  is mapped into itself and (after iterations) to  $\chi_+$ . Indeed, for any  $\chi$  in this interval,  $\chi_+ < \Pi_+(\chi) < \chi$  so that the sequence  $\{\Pi^p(\chi)\}$  converges to  $\chi_+$ . If  $\beta = \pi/2$ ,  $\chi_+ = \pi/2$ .

Consider next the interval  $\pi/2 < \beta < \beta_F(\hat{\Sigma} = -\pi)$  (see Fig. 14). Now  $\chi_+ > \pi/2$  and  $\tilde{\chi}_+ < \pi/2$ .

<sup>37</sup>At  $\beta = \beta_{1u}(\beta_{1d})$  the fixed point lies at the maximum (minimum) of  $\Pi(\chi)$ .

<sup>38</sup> $\Pi \circ \Pi \circ \dots \circ \Pi$  ( $p$  times)  $\equiv \Pi^p$ .



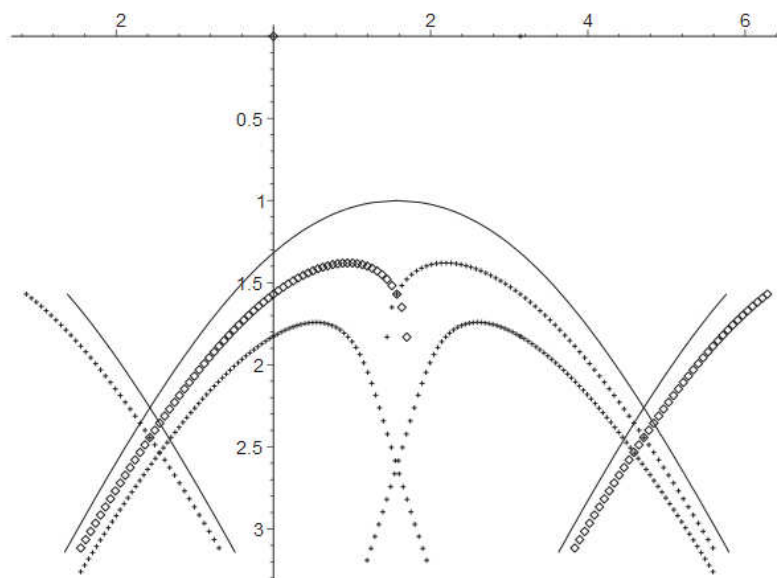


Fig. 14. The  $\beta$ -values for superstable orbits.

The iterates of any point in  $[0, \tilde{\chi}_+]$  reach at some stage the interval  $[\tilde{\chi}_+, \chi_+]$ . Indeed, the function  $\Pi_+(\chi)$  is monotonically increasing on  $[0, \tilde{\chi}_+]$  so that the sequence  $\{\chi, \Pi_+(\chi) \dots \Pi_+^p(\chi)\}$  increases monotonically until  $\Pi_+^p(\chi)$  gets larger than  $\tilde{\chi}_+$ . Then the following transformations under  $\Pi_+$  are obvious:

$$[\tilde{\chi}_+, \chi_+] \Rightarrow [\chi_+, \beta] \Rightarrow [\Pi_+(\beta), \chi_+]. \quad (\text{F.5})$$

Now, for  $\beta < \beta_{2u}$  ( $\hat{\Sigma} = -\Pi$ ),

$$\Pi_+(\beta) \equiv \beta \sin \beta > \frac{\pi}{2} \quad (\text{F.6})$$

(cf. (F.2)) so that the last interval is contained in  $[\pi/2, \chi_+]$ . Thus  $\Pi_+^2$  maps  $[\pi/2, \chi_+]$  into itself. The same is true for  $[\chi_+, \beta]$  which is mapped by  $\Pi_+$  into  $[\pi/2, \chi_+]$  and by  $\Pi_+^2$  into itself (cf. (F.5)). It is also true that, for  $\chi \in [\pi/2, \beta]$ :

$$(\Pi_+^2)'(\chi) = \Pi_+'(\Pi_+(\chi))\Pi_+'(\chi) > 0; \quad (\text{F.7})$$

because  $\beta < \pi$ . This inequality is true for all  $\beta \in [\pi/2, \beta_{2u}]$ . It follows from the property  $S(\Pi) < 0$ , (236) that  $\Pi_+^2$  has at most three fixed points in  $[\pi/2, \beta]$ .

We show now that, if  $\beta < \beta_F (< \beta_{2u})$  the only invariant set of  $\Pi_+^2$  in  $[\pi/2, \beta]$  is  $\{\chi_+\}$ . Indeed,  $\chi_+$  is the only root of  $\Pi_+^2(\chi) = \chi$  in  $[\pi/2, \beta]$  in this interval of values of  $\beta$ . For  $\chi = \pi/2$ ,  $\Pi_+^2(\chi) > \chi$  and for  $\chi = \beta$ ,  $\Pi_+^2(\chi) < \chi$ . Thus, for all  $\chi \in [\pi/2, \chi_+]$ ,  $\Pi_+^2(\chi) > \chi$  and for all  $\chi \in [\chi_+, \beta]$ ,  $\Pi_+^2(\chi) < \chi$ . It follows that for  $\chi \in [\pi/2, \chi_+]$  the sequence  $\{\chi, \Pi_+^2(\chi) \dots\}$  is monotonically increasing and can only converge to  $\chi_+$ .

Similarly  $\{\chi, \Pi_+^2(\chi) \dots\}$  is monotonically decreasing for  $\chi \in [\chi_+, \beta]$  and converges to  $\chi_+$ . Thus the only invariant set is indeed  $\{\chi_+\}$ .

We now turn to the situation  $\beta_F < \beta < \beta_{2u}$ . There are now three points  $(\chi_L, \chi_+, \chi_R)$  in  $[\pi/2, \beta]$  where  $\Pi_+^2(\chi) - \chi$  changes sign. Consequently, the sequences  $\{\chi, \Pi_+^2(\chi) \dots\}$  converge (monotonically increasing or decreasing) to  $\chi_L$  or  $\chi_R$  according to whether they start in  $[\pi/2, \chi_+]$  or in  $[\chi_+, \beta]$ . Now, the image  $\chi'_L$  of  $\chi_L$  under  $\Pi_+$  must be a fixed point of  $\Pi_+^2$  because  $\Pi_+^3(\chi_L) = \Pi_+^2(\chi'_L) = \Pi_+(\chi_L) = \chi'_L$  so that  $\chi'_L \equiv \chi_R$ . Thus  $\chi_+$  and an orbit of period two exhaust the invariant sets of  $\Pi_+$  for  $\beta < \beta_{2u}$ . The discussion of  $\Pi_-$  is totally symmetric to the above with  $\beta_{2u}$  replaced by  $\beta_{2d}$ .

To conclude, if  $\hat{\Sigma} = -\pi$ ,  $\beta_{2u} = \beta_{2d}$  and in the interval  $0 < \beta < \beta_{2u}$  the mapping  $\Pi$  has no other invariant sets apart from at most three fixed points and two orbits of period two.

### F.3. The general situation

The arguments concerning the contraction of intervals to the fixed points of  $\Pi$  and  $\Pi^2$  may all be taken over from the situation  $\hat{\Sigma} = -\pi$  of the previous section. We describe only the main features.

- (i) Assume  $\hat{\Sigma} = -\pi + \delta$ ,  $\delta < \pi/2$ . As  $\delta \rightarrow \pi/2$ ,  $\beta_{1u}(\hat{\Sigma}) \rightarrow 0$ ,  $\beta_{1d}(\hat{\Sigma}) \rightarrow \pi$ ,  $\beta_{2u}(\hat{\Sigma}) \rightarrow \pi/2$ ,  $\beta_{2u}(\hat{\Sigma}) < \beta_{2d}(\hat{\Sigma})$  and  $\beta_S$  increases. (a) If  $\beta < \beta_S(\hat{\Sigma})$  (cf. Eq. (237)), there exists a single fixed

point  $\chi_+ > 0$ ; if further  $\beta < \beta_{1u}(\hat{\Sigma}) \equiv \pi/2 - \delta$ , then  $\tilde{\chi}_+ < \chi_+$  and all points in  $[-\beta, \beta]$  are attracted to  $\chi_+$ . If  $\beta_S(\hat{\Sigma}) > \beta > \beta_{1u}$ , but  $\beta < \beta_{2u}$ , then  $\tilde{\chi}_+ < \chi_+$  and under  $\Pi$ :

$$\begin{aligned} [-\beta, \beta] &\Rightarrow [\tilde{\chi}_+, \chi_+] \Rightarrow [\chi_+, \beta] \\ &\Rightarrow [\Pi(\beta), \chi_+] \subset \left[\frac{\pi}{2} - \delta, \chi_+\right] \end{aligned}$$

since  $\beta_{2u} \equiv \Pi(\beta) > \beta_{1u} = \pi/2 - \delta$ . As before, this implies  $(\Pi^2)' > 0$  on  $(\pi/2 - \delta, \beta)$ . For  $\beta_F < \beta < \beta_{2u} < \beta_S$  there appear two further fixed points of  $\Pi^2$ , to which the intervals on the left and right of  $\chi_+$  are contracted under  $\Pi^2$ . (b) If  $\beta > \beta_S(\hat{\Sigma})$  there are two further fixed points  $\chi_- < \chi_0 < 0$ ,  $\chi_-$  is stable,  $\chi_0$  unstable. The interval  $[\chi_0, \tilde{\chi}_+]$  is mapped eventually into  $[\tilde{\chi}_+, \chi_+]$  and further evolution is the same as above. If  $\beta < \beta_{1d} = \pi/2 + \delta$ ,  $\tilde{\chi}_- < \chi_-$  and the interval  $(\chi_-, \chi_0)$  is contracted under  $\Pi$  to  $\chi_-$ . If  $\beta_{2d} > \beta > \beta_{1d}$ ,  $\tilde{\chi}_- > \chi_-$  and (possibly after iteration)

$$\begin{aligned} [\tilde{\chi}_-, \chi_0] &\Rightarrow [\chi_-, \tilde{\chi}_-] \Rightarrow [-\beta, \chi_-] \\ &\Rightarrow [\chi, \Pi(-\beta)] \subset \left[\chi_-, -\frac{\pi}{2} - \delta\right]. \end{aligned}$$

The last inclusion follows from  $\beta_{2d} > -\Pi(-\beta) > \beta_{1d}$ . A consequence of this inclusion is that  $(\Pi^2)'(\chi) > 0$  for  $\chi \in [\chi_-, -\pi/2 - \delta]$ , similarly to (F.7). If  $\beta < \beta_F$ , one concludes that the only invariant set in  $[-\beta, \chi_0]$  is  $\{\chi_-\}$ . If  $\beta > \beta_F$ , a stable orbit of period two appears,

but no other invariant sets. Thus, for  $\hat{\Sigma} \in [-\pi, -\pi/2]$ , the invariant set of  $\Pi$  consists of at most three fixed points and two orbits of period two. The invariant set depends on the value of  $\hat{\Sigma}$  (as  $\delta$  approaches  $\pi/2$ , the “lower” orbit of period two disappears).

- (ii) If  $\hat{\Sigma} \in [-\pi/2, 0]$ , one verifies that  $\beta_{2u}(\hat{\Sigma}) < \pi/2$  and  $\beta_{2u}(-\pi/2) = \beta_{2u}(0) = \pi/2$  (cf. Fig. 14); also  $\beta_{2u} \leq \beta_{2d}$  with equality at  $\hat{\Sigma} = 0$ . Further,  $\beta_S(\hat{\Sigma}) > \pi/2$  so that there is no fixed point in  $\chi < 0$ . For all  $\beta < \beta_{2u}$ , there is only one fixed point at  $\chi_+ > 0$ . Let  $\hat{\chi} \equiv \max[-\beta, \tilde{\chi}_+]$ , ( $\tilde{\chi}_+ < 0$ ). The image under  $\Pi$  of  $[\hat{\chi}, \chi_+]$  is  $[\chi_+, \beta]$ . The latter is mapped back onto  $(\Pi(\beta), \chi_+) \subset [-\pi/2 - \hat{\Sigma}, \chi_+]$ . Since  $\beta < \pi/2 - \hat{\Sigma}$  (the latter is the position of the minimum), it follows as before that  $(\Pi^2)' > 0$  on  $[-\pi/2 - \hat{\Sigma}, \beta]$ . This interval may thus contain one or three fixed points of  $\Pi^2$ . Repeating the argument of the previous situation, we conclude that the invariant sets of  $\Pi$  consist of at most one fixed point and one orbit of period two.
- (iii) If  $\hat{\Sigma} \in [0, \pi/2]$ , the situation is totally symmetrical to the previous one, with  $\beta_{2d}$  now interchanged with  $\beta_{2u}$  (cf. Fig. 14) and the unique fixed point of  $\Pi$  being now situated at  $\chi_- < 0$ . The invariant set of  $\Pi$  consists of at most one fixed point and one 2-orbit.
- (iv) If  $\hat{\Sigma} \in [\pi/2, \pi]$ , the situation is symmetrical (in the sense above) to that in (i), with the same conclusion. This ends the justification of Lemma 7.2.