THE DUFFING EQUATION AT LARGE FORCING AND DAMPING

I. S. Stefanescu

Institut für Theoretische Kernphysik

Universität Karlsruhe,Karlsruhe

Germany

The paper presents a proof that the Duffing equation:

$$\ddot{y} + 2\Delta\dot{y} + y^3 = \Gamma\cos t$$

admits of an infinite sequence of bifurcation curves in the Γ-Δ plane, alternately of the saddle - node and odd - simply 2π-periodic type, whose maxima lie at large Γ along the line:

$$\Delta_c(\Gamma) = \frac{1}{12\pi}\ln\Gamma - \frac{1}{3\pi}\ln\ln\Gamma + C_o$$

with a constant C_o given in the text. The positions of the maxima are interlaced in asymptotically equal intervals of $\Gamma^{1/3}$, with a spacing of 1.403 units. For $\Delta > \Delta_c(\Gamma)$, the Duffing equation admits of a unique periodic solution if Γ is large enough.

I. Introduction

The Duffing equation:

$$\ddot{y} + 2\Delta\dot{y} + y^3 = \Gamma\cos\bar{t} \qquad , \Delta,\Gamma > 0 \tag{1.1}$$

is a classical nonlinear equation: it exhibits a large variety of periodic solutions, not necessarily of the same period as the driving force, and whose number and appearance changes with the values of the parameters Δ,Γ . These solutions have been well studied by numerical or approximate analytic methods (see e.g. Refs.[1-7]). Knowledge about the periodic solutions of (1.1) is presented in a plot in the $\Delta-\Gamma$ plane (cf. Refs.[5-7]) of the boundaries of the domains where (1.1) admits of a certain type of solutions (e.g. with a given period $2\pi\, m/n$). The appearance of these plots is increasingly complicated, as the damping is decreased (see, e.g. Fig.1 of Ref.[5]). If Δ is large enough, at every fixed Γ , the situation simplifies: the equation admits of a unique periodic solution. We notice that, if (1.1) has only one periodic solution $y_p(t)$ at some Δ,Γ,then $y_p(t)$ has period 2π and its Fourier series contains only odd harmonics. Indeed, $y_{p1}(t) = -y_p(t+\pi)$ is also a periodic solution of (1.1) and it is, by assumption, identical to $y_p(t)$.

Assume the damping Δ is neither too large, nor too small (at small Γ , e.g. $0.1 \leq \Delta \leq 0.5$) and that we increase Γ gradually, starting from $\Gamma = 0$, where the equation has the unique periodic solution $y_p = 0$. According to the $\Delta-\Gamma$ plots of Refs.[5-7] (see, e.g. Fig.3 of Ref.[6]) we meet in this process a sequence of bifurcation curves from $y_p(t)$ either of the saddle - node type : $\Delta = \Delta_{SN}^{(p)}(\Gamma)$ or of the flip (odd periodic - simply periodic) type: $\Delta = \Delta_F^{(p)}(\Gamma)$. If Δ is not too small, these curves are well separated and their type alternates; e.g. $\Delta = \Delta_{SN}^{(p)}(\Gamma)$ intersects the line Δ = const in two points with abscissas $\Gamma_{L,SN}^{(p)}$, $\Gamma_{R,SN}^{(p)}$ and has between them a clear maximum $\Delta_{SN}^{(p)}$ at $\Gamma_{SN}^{(p)}$; the analogous statement is true for $\Delta_F^{(p)}(\Gamma)$. When traversing a saddle - node bifurcation curve, $y_p(t;\Gamma)$ can be continued through the left boundary point $\Gamma_{L,SN}^{(p)}$ up to $\Gamma_{R,SN}^{(p)}$, where it annihilates with an (unstable) solution originating at $\Gamma_{L,SN}^{(p)}$. In a small interval $[\Gamma_{R,SN}^{(p)}, \Gamma_{L,F}^{(p)}]$, the stable solution created at $\Gamma_{L,SN}^{(p)}$ is the unique solution $y_p(t)$ of (1.1). When we traverse a flip bifurcation curve, $y_p(t)$ loses its stability and gives rise, for $\Gamma > \Gamma_{L,F}^{(p)}$ to two stable,simply periodic solutions of (1.1). If Δ is small enough, the latter cascade at an increase of Γ through period doubling bifurcations to an attracting chaotic motion, presented in well-known pictures in Ref.[5]. If Γ is further increased, the process is reversed and $y_p(t)$ regains its stability and uniqueness in a small interval $[\Gamma^{(p)}_{R,F}, \Gamma^{(p+1)}_{L,SN}$ - Numerical evidence suggests that the sequence of bifurcation curves $\Delta^{(p)}_{SN}(\Gamma)$, $\Delta^{(p)}_{F}(\Gamma)$ is infinite; the positions $\Gamma^{(p)}_{SN}, \Gamma^{(p)}_{F}$ of their maxima appear to be equidistant in the variable $\Gamma^{1/3}$ (Refs.[6,11]).

Now, the literature contains no explanation from first principles of this state of facts; in particular, with one exception (Ref.[12]), there exists no description of the domain of values in the $\Delta - \Gamma$ plane where eqn. (1.1) admits of a unique periodic solution. In Ref.[12], W. S. Loud shows, using a result of Cartwright and Littlewood (Ref.[13]) that, if an harmonic term +ky is present in (1.1), then (1.1) has a unique periodic solution at every fixed Γ, provided Δ is large enough (essentially $\Delta >$ const$\cdot\Gamma$); the theorem of Ref.[13] is, however, not readily extensible to the situation k = 0 and the limitation on Δ is weak at higher Γ.

In this paper, we consider the case when both Δ, Γ are large; it turns out that the solution of (1.1) can be approximated in a controlled manner in this range of parameters so that we obtain a nontrivial expression for the halfperiod (i.e. $t \to t + \pi$) Poincaré mapping $\mathbb{P}(\Gamma;\Delta)$ pertaining to (1.1). This allows us both to settle the question of uniqueness and to prove that, indeed, an infinite sequence of bifurcation curves of alternating types does occur; these are the natural continuation of those observed on the computer (Refs.[1,4-7]) in an intermediate range of values of Γ. More precisely, the results are as follows: there exists in the $\Delta - \Gamma$ plane a curve:

$$\Delta = \Delta_c(\Gamma) = \frac{1}{12\pi}\ln\Gamma - \frac{1}{3\pi}\ln\ln\Gamma + C_o + O\left(\frac{\ln\ln\Gamma}{\ln\Gamma}\right) \qquad (1.2)$$

where C_o is given in eqn. (8.2) below, so that, if $\Delta > \Delta_c(\Gamma)$, eqn. (1.1) admits of a unique periodic solution, provided Γ is large enough. The (unique) maxima $\Delta^{(p)}_{SN}, \Delta^{(p)}_{F}$ of the bifurcation curves $\Delta^{(p)}_{SN}(\Gamma)$, $\Delta^{(p)}_{F}(\Gamma)$ interlace and lie asymptotically on the curve (1.2); their positions $\Gamma^{(p)}_{SN}, \Gamma^{(p)}_{F}$ are asymptotically equidistant in the variable $\Gamma^{1/3}$, with a spacing:

$$\delta(\Gamma^{1/3}) = \Gamma_F^{(p)\,1/3} - \Gamma_{SN}^{(p)\,1/3} = \Gamma_{SN}^{(p+1)\,1/3} - \Gamma_F^{(p)\,1/3} \sim 1.403 \sim \frac{\pi}{\sqrt{3}\int_{-\pi/2}^{\pi/2}|\sin t|^{1/3}dt} \tag{1.3}$$

In a comparison with numerical observations, the somewhat unexpected result is the logarithmic increase of the maxima of the bifurcation curves, which has not yet been noticed. From a theoretical point of view, it has the consequence that for large Γ and Δ near the curve (1.2), the Poincaré map $\mathbb{P}$ of (1.1) contracts phase space indefinitely so that, as far as its periodic points (corresponding to periodic solutions of (1.1)) are concerned, it is asymptotically equivalent to a onedimensional mapping; the latter turns out to be simply:

$$\Pi : \chi \longrightarrow \beta\cos(\chi+\Sigma) \qquad (\mathrm{mod}\ 2\pi) \tag{1.4}$$

which maps the circle S^1 into itself, with β, Σ known functions of Γ, Δ. The largest part of the paper is devoted to a derivation (and a discussion) of this limit; once it is established, the statements above on bifurcation curves are simple consequences.

We recall that a study of the periodic solutions of eqn. (1.1) and of their bifurcations at large Γ with Δ held constant is given in two papers (Refs.[12,13]) by J. G. Byatt - Smith. This limiting situation requires also extensive numerical work. Several results of Refs. [12,13] concerning the asymptotic expansions of special solutions of (1.1) appear also in Sections V, VI of this paper, although obtained in a different manner.

In many studies, eqn. (1.1) is supplemented by an harmonic term +ky. The situation $k<0$ leads, for small Γ, Δ to a chaotic motion that may be understood to a large extent analytically (see Ref.[8], §4.3,5.3). If the coefficient k is held fixed with increasing Γ, it turns out that the description of the limiting situation considered in this paper is independent (to leading order in $1/\Gamma$) of its precise value. A short discussion of the changes appearing if $k \neq 0$ is given in the summary.

We introduce next the main notation, together with some comments on the properties of (1.1) at large Γ. Define first:

$$x = y/\Gamma^{1/3}, \quad t = \bar{t} - 3\pi/2, \quad \varepsilon = 1/\Gamma^{2/3}, \quad \mu = \Delta/\Gamma^{2/3} \tag{1.5}$$

so that eqn. (1.1) becomes:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + x^3 = \sin t \tag{1.6}$$

and $\Gamma \to \infty$ means $\varepsilon \to 0$. Eqn. (1.6) is the form of Duffing's equation used throughout this paper.

Assume now that $\Delta = \Delta(\Gamma)$ is a monotonically increasing function of Γ. If, as $\Gamma \to \infty$, $\mu = \Delta/\Gamma^{2/3} > \mu_o > 0$, we change variables further in (1.6) to:

$$z = \mu x\ , \quad \bar{\varepsilon} = \varepsilon/\mu \quad , \quad \bar{\mu} = 1/\mu^3 \tag{1.7}$$

we obtain:

$$\bar{\varepsilon} \ddot{z} + 2\dot{z} + \bar{\mu} z^3 = \sin t \tag{1.8}$$

As $\varepsilon \to 0$, it reduces to:

$$2\dot{z} + \bar{\mu} z^3 = \sin t \tag{1.9}$$

It is easy to show that, if $\bar{\mu}$ is bounded, eqn. (1.9) admits of a unique periodic solution which can be improved by straightforward iteration of (1.8) to a periodic solution $z_p(t)$ of the latter; further, $z_p(t)$ is unique (see Sect. IX).

However, if $\mu \to 0$ as $\varepsilon \to 0$, eqn. (1.6) reduces in this limit to:

$$x^3 = \sin t \tag{1.10}$$

with the solution:

$$x_{oo}(t) = (\sin t)^{1/3} \tag{1.11}$$

Corrections to $x_{oo}(t)$ cannot be obtained by iterating eqn. (1.6), since the derivatives of $x_{oo}(t)$ at $t = 0$ are not finite. We expect nevertheless (1.11) to be a good approximation to periodic solutions of (1.6) away from $t = 0$. The depertures of the solutions of (1.6) from (1.11) near $t = 0$ are obtained by a boundary layer analysis: let:

$$t = \mu^{3/5}\tau \quad\quad , \; x = \mu^{1/5}\eta \tag{1.12}$$

so that (1.6) becomes:

$$\frac{\varepsilon}{\mu^{8/5}} \frac{d^2\eta}{d\tau^2} + 2\frac{d\eta}{d\tau} + \eta^3 = \mu^{-3/5} \sin(\tau\mu^{3/5}) = \tau - \frac{\mu^{6/5}}{6}\tau^3 + \dots \tag{1.13}$$

To zeroth order in $\mu^{6/5}$, we are interested in that solution of (1.13) which behaves like $\tau^{1/3}$ as $\tau \to -\infty$, so that it matches $x_{oo}(t)$. If $\varepsilon/\mu^{8/5} \to 0$ as $\varepsilon \to 0$ (i.e. $\Delta/\Gamma^{1/4} \to \infty$), this solution is obtained by perturbing that of:

$$2\frac{d\eta}{d\tau} + \eta^3 = \tau \tag{1.14}$$

If, however, $\Delta/\Gamma^{1/4} \to 0$ as $\varepsilon \to 0$, the appropriate boundary layer quantities are:

$$t = \varepsilon^{3/8}\tau , \quad x = \varepsilon^{1/8}\eta , \quad \gamma = \mu/\varepsilon^{5/8} \ (= \Delta/\Gamma^{1/4}) \tag{1.15}$$

in terms of which (1.6) becomes:

$$\frac{d^2\eta}{d\tau^2} + 2\gamma \frac{d\eta}{d\tau} + \eta^3 = \varepsilon^{-3/8} \sin(\varepsilon^{3/8}\tau) \sim \tau - \varepsilon^{3/4}\tau^3/6 + \ldots \tag{1.16}$$

As $\varepsilon \to 0$, the solutions of (1.16) obeying $\eta(\tau) \sim \tau^{1/3}$ as $\tau \to -\infty$ are oscillatory as $\tau \to +\infty$ and are damped in a "time" $\tau \sim 1/\gamma \to \infty$.

We distinguish thus three regimes of (1.6) for large Γ : (i) $\mu > \mu_o > 0$ as $\varepsilon \to 0$; (ii) $\mu < \mu_o$ and $\varepsilon/\mu^{8/5} <$ const as $\varepsilon \to 0$; (iii) $\mu \to 0$, $\gamma = \mu/\varepsilon^{5/8} < \gamma_o$ as $\varepsilon \to 0$. We shall show that situations (i), (ii) lead to unique periodic solutions of (1.6) for Γ large; the transition to nonuniqueness occurs in (iii) (cf. eqn. (1.2)). Since the latter is the main concern of this paper, we shall assume throughout (except for Section IX) that $\gamma = \Delta/\Gamma^{1/4} < \gamma_o$ as $\varepsilon \to 0$ and use the notation (1.15). In view of (1.2), we find it convenient to use in the domain (iii) instead of μ the variable:

$$k = \frac{\mu}{\varepsilon \ln \frac{1}{\varepsilon}} = \frac{3}{2}\frac{\Delta}{\ln \Gamma} \tag{1.17}$$

so that bifurcations occur when $k \sim 1$ as $\varepsilon \to 0$.

The paper is organized as follows: in Section II, we give general preparatory statements on the solutions of (1.6) and on the way they approach each other. Section III introduces the inner and outer expansions associated to (1.6); these are combined and improved to two special, nonoscillatory solutions $X_L(t)$, $X_R(t)$ of (1.6), defined for $t < 0$, $t > 0$ in turn. These solutions are taken as references for $t < 0$, $t > 0$ and the Poincaré map is defined in terms of the differences:

$$v_L(t) = x(t) - X_L(t) , \ t < 0 \ ; \ v_R(t) = x(t) - X_R(t) , \ t > 0 \tag{1.18}$$

In Section IV, we give a precise bound, depending on ε , on the region $D(\varepsilon)$ of phase space where invariant sets of the map $\mathbb{P}$ may exist. Sections V and VI establish a controlled approximate expression for $\mathbb{P}(\varepsilon;k)$. Section VII discusses the limiting form $\mathbb{J}$ of $\mathbb{P}(\varepsilon;k)$ as $\varepsilon \to 0$, eqn. (1.4), and Section VIII the extent to which the properties of $\mathbb{J}$ may be transferred to statements on $\mathbb{P}(\varepsilon;k)$ for small, nonzero ε . In particular, we establish the announced properties concerning the bifurcation lines of $\mathbb{P}$ and the uniqueness of the solution for $\Delta_c(\Gamma) < \Delta <$ const $\cdot \Gamma^{1/4}$. Finally, Section IX proves the uniqueness of the periodic solution in the asymptotic domains (i),(ii) of parameter values. An informal discussion of the results is given in Section X.

Refs.[14,15] contain a more detailed treatment of some items (in particular of Section IX)

II. General preparation

<u>Lemma 2.1</u> : There exists a rectangle:

$$D : |x| < B_1 \quad , \quad \left|\frac{dx}{dt}\right| < \frac{B_2}{\sqrt{\varepsilon}} \tag{2.1}$$

so that all solution paths $(x(t), \dot{x}(t))$ of (1.6) eventually get inside it. The constants B_1, B_2 are independent of ε, μ if ε and ε/μ are sufficiently small.

<u>Proof</u>: Assume $\mu/\sqrt{\varepsilon} = \Delta/\Gamma^{1/3} <$ const, as $\varepsilon \to 0$ and consider the (Liapunov) function $\Phi(p;x;t)$ given by: $(p = dx/dt)$

$$\Phi(p;x;t) = \exp\left[-L(p;x;t)\right] \tag{2.2}$$

$$L(p;x;t) = E(p;x;t) + D(p;x;t) \tag{2.3}$$

$$E(p;x;t) = \frac{p^2}{2} + \frac{x^4}{4} - x \sin t \tag{2.4}$$

$$\begin{aligned} D(p;x;t) &= 0 && \text{if } p > \max\left[(|x|/\mu)^{1/2}, (A/\mu)^{1/2}\right] \\ &= \varepsilon(p - (x/\mu)^{1/2}) && \text{if } |p| < (|x|/\mu)^{1/2}, \; x > A \\ &= -2\varepsilon(x/\mu)^{1/2} && \text{if } p < -(x/\mu)^{1/2}, \; x > A \\ &= -2\varepsilon(x/A)(|x|/\mu)^{1/2} && \text{if } p < -(A/\mu)^{1/2}, |x| < A \\ &= 2\varepsilon(|x|/\mu)^{1/2} && \text{if } p < -(|x|/\mu)^{1/2}, \; x < -A \\ &= -\varepsilon(p - (|x|/\mu)^{1/2}) && \text{if } |p| < (|x|/\mu)^{1/2}, \; x < -A \end{aligned} \tag{2.5}$$

Differentiation of (2.2) and use of (1.6) establishes that, provided ε and ε/μ are sufficiently small: (a) $\Phi \to 0$ as $|x|, |p| \to \infty$, uniformly in all directions and with respect to t, for all t; (b) $d\Phi/dt(p;x;t) > \delta > 0$ for all t outside a rectangle D_1: $|x| < A, |p| < (A/\mu)^{1/2}$, e.g. for $A > 3$; (c) $\Phi > 0$ outside D_1. By a well known theorem (Ref.[16], p.371, ch.VII, §3), there exists under hypotheses (a), (b), (c) on Φ a rectangle (2.1), containing D_1 in its interior, so that all solution paths eventually come into it. The parameters of the rectangle D, eqn.(2.1) may be inferred from those of (2.1) as follows: let:

$$\Phi_{o1} = \min_t \min_{p,x \in \partial D_1} \Phi(p;x;t) \tag{2.6}$$

and choose D so that:

$$\Phi_{o2} = \max_t \max_{p,x \in \partial D} \Phi(p;x;t) \tag{2.7}$$

obeys $\Phi_{o2} < \Phi_{o1}$. One verifies that $B_1 \simeq 3$, $B_2 \simeq 9$ satisfy this.

If $\Gamma^{1/3}/\Delta > 1$, a function Φ with properties (a),(b),(c) may be taken directly from Ref.[16], p. 377 (with obvious changes), in an example due to G. Reuter (Ref.[17]). This ends the proof.

Comment 2.1: The boundary line $\Delta/\Gamma^{1/3} \sim 1$ appearing in the proof of Lemma 2.1 runs inside the asymptotic domain (ii) of the parameters ε, μ of the Introduction. It plays a role also below.

The following places a bound on the manner in which two solutions $x_o(t)$, $x(t)$ of (1.6) approach each other, once they are in D. For simplicity assume that, if $\mu < A$, eqn. (1.6) admits of solutions $x_o(t;\varepsilon)$ which stay in D for $t > t_o(\varepsilon)$ and obey:

(H1) There exist $a,b > 0$, independent of ε, so that $|x_o(t;\varepsilon)| > a$, $|dx_o(t;\varepsilon)/dt| < b$, for $t \in [t_1,t_2]$, with $0 < t_1 < t_2 < \pi \pmod{\pi}$, $t > t_o(\varepsilon)$.

It is easy to obtain such solutions (see Section III). With this, for any other $x(t)$, staying in D for $t > t_o(\varepsilon)$, we may state:

Lemma 2.2: Assume $\mu/\varepsilon^{1/2} = \Delta/\Gamma^{1/3} < A$ and $\varepsilon/\mu = 1/\Delta \to 0$ as $\varepsilon \to 0$. Then, for ε sufficiently small, there exist constants k, C, independent of ε, so that, for $t \in [t_1,t_2]$:

$$\max\left[|x(t) - x_o(t)|, \varepsilon^{1/2}|dx/dt(t) - dx_o/dt(t)|\right] < K \exp[-C\mu(t-t_1)/\varepsilon] \tag{2.8}$$

Proof: Let $C < 1$ and:

$$u(t) = (x(t) - x_o(t)) \exp[C\mu(t-t_1)/\varepsilon] \quad ; \tag{2.9}$$

it verifies:

$$\varepsilon\ddot{u} + 2\mu\dot{u}(1-C) + u\left[3x_o^2(t) + (C^2 - 2C)\mu^2/\varepsilon\right] + 3u^2 \exp[-C\mu(t-t_1)/\varepsilon]x_o(t)$$
$$+ u^3 \exp[-2C\mu(t-t_1)/\varepsilon] = 0 \tag{2.10}$$

Consider the Liapunov function:

$$L_u = \frac{1}{2}(\varepsilon\dot{u} + 2\mu\bar{C}u)^2 + \varepsilon G(u,t) \tag{2.11}$$

with $\bar{C} = 1 - C$ and:

$$G(u,t) = \int_0^u u' F(u',t)\, du' \tag{2.12}$$

where $uF(u,t)$ denotes the last three terms of (2.10). The forms $F(u,t)$, $G(u,t)$ are positive definite for $t \in [t_1,t_2)$ if $C < a^2/(8A^2)$. Using (2.10) we obtain:

$$\frac{dL_u}{dt} = -2\mu\bar{C}\left(u^2F(u,t) - \frac{\varepsilon}{2\mu\bar{C}}\frac{\partial G}{\partial t}\right) = -2\mu\bar{C}\,u^2H(u,t) \tag{2.13}$$

It is easy to verify that, if, e.g. $C < \min(3/4,\ a^2/(8A^2))$ and $\varepsilon/\mu = 1/\Delta$ is small enough, then $H(u,t)$ is positive definite for $t \in [t_1,t_2)$. Thus, the solution paths $(u,\dot{u})$ of (2.10) stay contained in the bounded domain:

$$L_u(t) < L_u(t_1) \tag{2.14}$$

for $t \in [t_1,t_2]$. But $L_u(t_1) = O(\varepsilon)$; therefore $G(u,t) = O(1)$ for $t \in [t_1,t_2]$ and, since $G(u,t) = u^2 F_1(u,t)$ with F_1 strictly positive definite, it follows that $u = O(1)$ for $t \in [t_1,t_2]$. Further, since $\varepsilon\dot{u} + 2\mu\bar{C}u = O(\varepsilon^{1/2})$, we obtain $u = O(1/\varepsilon^{1/2})$ for $t \in [t_1,t_2]$. Returning to (2.9), we obtain (2.8).

Comment 2.2: A similar statement is valid if $B\varepsilon^{1/2} < \mu < A$ $(B\Gamma^{1/3} < \Delta < A\Gamma^{2/3})$ but we shall not need it below.

Comment 2.3 : With the notation (1.17), the difference $v(t) = x(t) - x_o(t)$ obeys eventually on $[t_1,t_2]$ (mod 2π):

$$|v(t)| < K\,\varepsilon^{kC(t-t_1)} \quad , \quad |dv/dt| < K\,\varepsilon^{-1/2}\,\varepsilon^{kC(t-t_1)} \tag{2.15}$$

so that, if $k = O(1)$ and sufficiently small, although $|v(t)| \to 0$ as $\varepsilon \to 0$, this may not be true for the derivatives. This may indicate why the asymptotic "line" $k \sim 1$ plays a special role.

From Section III up to Sect. VIII we shall only be concerned with the situation $\Delta < C\,\Gamma^{1/4}$.

III. Inner and outer expansions. Existence of some special solutions ($\Delta < C\Gamma^{1/4}$)

If t lies away from $n\pi$, we may iterate eqn. (1.6) formally, starting with (1.11); we expect that, for small ε, μ and $|t| > 0$ (mod π) a solution of (1.6) exists close to the formal expansion (the "outer expansion"):

$$x_o(t) = \sum_{k,l \geq 0} \mu^k\,\varepsilon^l\,x_{kl}(t) \tag{3.1}$$

with $x_{oo}(t)$ of (1.11),

$$x_{1o}(t) = -2\,\dot{x}_{oo}(t)/\,3\,x_{oo}^2 \quad , \text{ etc.} \tag{3.2}$$

In general, we may state:

<u>Lemma 3.1</u>: $$x_{kl}(t) = t^{1/3 - 5k/3 - 8l/3} \sum_q a_{klq} t^{2q} \tag{3.3}$$

where the sum is uniformly and absolutely convergent on $[-\pi+\sigma, \pi-\sigma]$, for any $\sigma > 0$.

The proof is straightforward by induction. We write in the following $x_o^{(K,L)}$ for (4.1) with the sum restricted to $k \leq K$, $l \leq L$.

If $\Delta < C\Gamma^{1/4}$, a formal "inner" expansion may be obtained from (1.15), (1.16) as:

$$x_{iL}(t) = \varepsilon^{1/8} \eta_L(t) = \varepsilon^{1/8} \sum_q \varepsilon^{3q/4} \quad \eta_{qL}(\tau; \gamma) \tag{3.4}$$

where the $\eta_{qL}(\tau; \gamma)$ are those solutions behaving in turn like $\tau^{2q+1/3}$ as $\tau \to -\infty$ of the equations:

$$\frac{d^2\eta_o}{d\tau^2} + 2\gamma \frac{d\eta_o}{d\tau} + \eta_o^3 = \tau \tag{3.5}$$

$$\frac{d^2\eta_1}{d\tau^2} + 2\gamma \frac{d\eta_1}{d\tau} + 3\eta_o^2 \eta_1 = -\frac{\tau^3}{6} \quad ,\text{etc.} \tag{3.6}$$

Concerning (3.4), we have the following:

<u>Lemma 3.2</u>: Expansion (3.4) is well defined, i.e. the solutions $\eta_{qL}(\tau)$ occurring in it exist and are unique. The asymptotic expansion of the $\eta_{qL}(\tau)$ as $\tau \to -\infty$ is given by:

$$\eta_{qL}(\tau) \sim \tau^{2q+1/3} \sum_{k,\ell} a_{klq} \; \tau^{-5k/3 - 8l/3} \; \gamma^k \tag{3.7}$$

where a_{klq} are <u>the same</u> constants as those of (3.3).

The proof is given in Appendix A. We denote $x_{iL}^{(Q)}$ the sum in (3.4) limited to $q \leq Q$.

<u>Comment 3.1</u>: For τ large positive, the solutions of (3.5),(3.6) are oscillatory and are damped on a τ- time scale $1/\gamma \gg 1$, so that there is a marked difference between the behaviour of the $\eta_{qL}(\tau)$ for $\tau \to -\infty$ and $\tau \to +\infty$. This justifies the index "L" in (3.4).

With the help of the finite versions of (3.1) and (3.4), we set up an approximant to a special solution of (1.6): consider to this end (Refs.[18,19])

$$x_{aL}(t) = \chi_{oL}(t; \varepsilon^\alpha) \, x_o^{(K,L)}(t) + \chi_i(t; \varepsilon^\alpha) \, x_{iL}^{(Q)}(t) \tag{3.8}$$

where $0 < \alpha < 3/8$, $\chi_{oL}(t;\varepsilon^\alpha)$ is of class C^2, supported on $[-\pi/2, -a\varepsilon^\alpha]$ and $\chi_{oL}(t;\varepsilon^\alpha) = 1$ on $[-\pi/2, -b\varepsilon^\alpha]$, $0 < a < b$; for $-\pi/2 < t < 0$:

$$\chi_i(t;\varepsilon^\alpha) = 1 - \chi_{oL}(t;\varepsilon^\alpha) \tag{3.9}$$

Now, $x_{aL}(t)$ is not a solution of Duffing's equation on $-\pi/2 < t < 0$ but is uniformly very close to a solution, for small ε, μ : indeed:(i) $x_o^{(K,L)}(t)$ verifies (1.6) up to terms of $O((\varepsilon/t^{8/3})^{L+1} + (\mu/t^{5/3})^{K+1})$ (cf. Lemma 3.1); (ii) $x_i^{(Q)}(t)$ verifies (1.16) up to terms of $O(\varepsilon^{3(Q+1)/4+1/8}\,\tau^{(2Q+3)/3})$; (iii) on the interval $[-b,-a]\,\varepsilon^\alpha$, the difference $x_i^{(Q)}(t) - x_o^{(K,L)}(t)$ is of $O(\varepsilon^{cP})$, with $P = \min(K,L,Q)$; $c>0$ this is a consequence of the identity of the coefficients a_{klq} in the expansions (3.3),(3.7), as asserted by Lemma 3.2. To express this quantitatively, let:

$$\mathcal{D}(x) = \varepsilon\ddot{x} + 2\mu\dot{x} + x^3 - \sin t \tag{3.10}$$

denote the action of the "Duffing operator" on functions of class C^2. With the help of the above observations (see Refs.[14,31] for calculational details), we can state:

Lemma 3.3: There exist constants $c_o, c_1 > 0$, independent of ε, μ, so that

$$\sup_{-\pi/2<t<0} |\mathcal{D}(x_{aL})(t)| < c_o\,\varepsilon^{c_1 P} \tag{3.11}$$

with $P = \min(K,L,Q)$.

To prove that a solution $X_L(t;K,L,Q;\varepsilon,\mu) \equiv X_L(t;\varepsilon)$ exists, approximated as close as we wish by $x_{aL}(t)$ for small ε, μ, provided only we take P sufficiently large, we proceed in a standard manner: let $r(t)$ be of class C^2 and obeying $r(-\pi/2) = dr/dt(-\pi/2) = 0$, write:

$$X_L(t) = x_{aL}(t) + r(t) \tag{3.12}$$

and show that the integral equation:

$$r(t) = -\frac{1}{\varepsilon}\int_{-\pi/2}^{t} \frac{v_1(t')\,v_2(t) - v_1(t)v_2(t')}{W(v_1,v_2)}\,(\mathcal{D}(x_{aL}) + 3\,x_{aL}r^2 + r^3)dt \tag{3.13}$$

admits of a solution in a sup - ball (including the first two derivatives) of radius $\varepsilon^{c_2 P}$, for some $c_2 > 0$. In (3.13), $v_1(t), v_2(t)$ are two independent solutions of the variational equation to (1.6) around $x_{aL}(t)$:

$$\varepsilon\ddot{v} + 2\mu\dot{v} + 3\,x_{aL}^2\,v = 0 \tag{3.14}$$

and W is their Wronskian.

To establish the statement on $r(t)$, eqn. (3.12), we need only a coarse bound on $v_1(t)$, $v_2(t)$ and their derivatives. We shall need in Section V solutions of similar equations, so we make below a more detailed statement about their behaviour.

Let to this end in (3.14):

$$v(t) = w(t)\exp[-\tfrac{\mu}{\varepsilon}(t+\pi/2)] \tag{3.15}$$

and define, with τ, γ of (1.15):

$$\Phi(\tau) \equiv (3x_{aL}^2 - \mu^2/\varepsilon)\varepsilon^{-1/4} \equiv 3\eta_{aL}^2 - \gamma^2 \tag{3.16}$$

Further, let $w_{1,2}^{(W)}(t)$ be the WKB functions:

$$w_{1,2}^{(W)}(\tau;\varepsilon) = \Phi^{-1/4}(\tau)\binom{\cos}{\sin}\Big[\int_\tau^{-\tau_0}\Phi^{1/2}(\tau')\,d\tau'\Big] \tag{3.17}$$

where $\tau_o > 0$ is chosen so that $\Phi > 0$ for $\tau < -\tau_o$. Since $\eta_L(\tau) \sim \tau^{1/3}$ for large $|\tau|$, such a choice of τ_o is always possible if $\Delta < C\Gamma^{1/4}$ ($\gamma < \gamma_o$). Notice: $|w_{1,2}^{(W)}(t)| \sim \varepsilon^{1/16}$ at $t = -\pi/2$ and ~ 1 at $|t| \sim \varepsilon^{3/8}$. Let now $v_{1,2}(t)$ be those solutions of (3.14) such that the corresponding $w_{1,2}(\tau)$ (by (3.15)) obey at $t = -\pi/2$ the same boundary conditions as $w_{1,2}^{(W)}(\tau)$ of (3.17). It is easy to verify (Refs.[20,21]) that the $w_{1,2}(\tau)$ are solutions of the integral equations:

$$w_{1,2}(\tau;\varepsilon) = w_{1,2}^{(W)}(\tau;\varepsilon) + \Phi^{-1/4}(\tau)\int_{-\frac{\pi}{2}\varepsilon^{-3/8}}^{\tau} R(\Phi)\,\Phi^{-1/4}(\tau')\,\sin\Big[\int_{\tau'}^{\tau}\Phi^{1/2}(\tau'')d\tau''\Big]w_{1,2}(\tau';\varepsilon)\,d\tau' \tag{3.18}$$

with:

$$R(\Phi) = \frac{5}{16}\Phi^{-3}\left(\frac{d\Phi}{d\tau}\right)^2 - \frac{1}{4}\frac{d^2\Phi}{d\tau^2}\Phi^{-2} \tag{3.19}$$

With the help of (3.7), we can establish $\Phi(\tau) \sim 3\tau^{2/3}$, $R(\Phi) \sim \tau^{-8/3}$; in these estimates, it is essential that $x_{aL}(t)$ is a smooth function, with bounded derivatives with respect to τ. Care is required if $\Phi(t)$ has oscillations. (see Ref.[14]). From (3.18), we can obtain bounds on $|w_{1,2}(\tau;\varepsilon) - w_{1,2}^{(W)}(\tau;\varepsilon)|$ and deduce:

<u>Lemma 3.4</u>: **If $-\varepsilon^{-\delta} < \tau < 0$, for δ sufficiently small, the solutions $w_{1,2}(\tau;\varepsilon)$ of (3.18) and their derivatives with respect to τ have a uniform limit as $\varepsilon \to 0$.**

The straightforward proof and estimates for the convergence are given in Appendix B.

Returning now to the main stream, i.e. to eqn. (3.13), we notice:

$$W(v_1,v_2) = \varepsilon^{-3/8}\exp[-2\tfrac{\mu}{\varepsilon}(t+\pi/2)] \tag{3.20}$$

so that we can bound the kernel of (3.13) uniformly on $[-\pi/2,0]$ by const$\cdot\varepsilon^{-5/8}$. Since $\sup|\mathcal{X}(x_{aL})(t)|$ can be made as small as one wishes, by letting P be large enough, eqn. (3.13) is a contraction of a ball of radius $\varepsilon^{c_2 P}$ into itself, for

c_2 sufficiently small. After evaluating the departure of the derivatives of $X_L(t)$ from those of $x_{aL}(t)$ (see Ref.[14] for details) we can state:

Theorem 3.1: Eqn. (1.6) admits of a solution $X_L(t;K,L,Q;\varepsilon)$ uniformly approximated together with its first two derivatives better than $\varepsilon^{c_2 P}$ on $[-\pi/2,0]$ by $x_{aL}(t)$, eqn. (3.8) and which obeys: $X_L(-\pi/2) = x_{aL}(-\pi/2)$, $dX_L/dt(-\pi/2) = dx_{aL}/dt(-\pi/2)$.

We turn next to the interval $\tau > 0$. We would like to obtain a solution analogous to $X_L(t)$ for $t > 0$, by matching the outer expansion (3.1) to a suitable inner expansion $\eta_R(\tau)$ of (1.16). The latter is not simply the continuation of $\eta_L(\tau)$, eqn. (3.4) (see Comment 3.1); further, all solutions of (3.5), (3.6) behave like $\tau^{2Q+1/3}$ as $\tau \to +\infty$, so that we cannot select unique terms by the boundary condition. Whereas the oscillations of most solutions $\eta_q(\tau)$ die out in a time $\tau \sim \gamma^{-1}$, there exist some for which the amplitude of the oscillations for $\tau < \gamma^{-1}$ is less than const$\cdot\gamma^{r+1}$, for any given $r > 0$. Indeed, look for a solution $\eta_{oR}(\tau)$ of (3.5) in the form:

$$\eta_{oR}(\tau) = \sum_{k=0}^{r} \eta_{ok}\gamma^k + \gamma^{r+1} v(\tau) \equiv \eta_o^{(r)}(\tau) + u(\tau) \tag{3.21}$$

where the $\eta_{ok}(\tau)$ are, in turn, the solutions behaving like $\tau^{1/3 - 5k/3}$ of:

$$\frac{d^2\eta_{oo}}{d\tau^2} + \eta_{oo}^3 = \tau \tag{3.22}$$

$$\frac{d^2\eta_{o1}}{d\tau^2} + 3\eta_{oo}^2\,\eta_{o1} = -2\frac{d\eta_{oo}}{d\tau}\ , \text{ etc.} \tag{3.23}$$

Clearly, since the damping terms are absent from (3.22-23), solutions with the required asymptotic behaviour (and corrections falling off sufficiently rapidly) are uniquely defined. The function $v(\tau)$ in (3.21) may be taken as any bounded solution (for $\tau > 0$) of:

$$\frac{d^2 v}{d\tau^2} + 2\gamma\frac{dv}{d\tau} + 3\eta_o^{(r)2} v + 3\eta_o^{(r)}\gamma^{r+1} v^2 + \gamma^{2r+2} v^3 = k(\tau) \tag{3.24}$$

where $k(\tau) \sim \tau^{1/3-5(r+1)/3}$ for large τ. It is easy to verify that such solutions exist, for γ small enough (cf. Lemma 4.3 below).

With this, the asymptotic behaviour of the $\eta_{oR}(\tau)$ is obtained as follows: for large τ :

$$\eta_{ok} \sim \sum_{1} a_{klo}\, \tau^{1/3 - 5k/3 - 8l/3} \qquad (3.25)$$

so that, for $\tau > \tau_o$, sufficiently large, but independent of ε, it is true that

$$| \eta_{oR} - \sum_{k \leq L} \gamma^k \eta_{ok,L} | \leq C(L)(\gamma^{r+1} + \tau^{1/3-5(L+1)/3}) \qquad (3.26)$$

where $\eta_{ok,L}$ is an L - truncation of the sum in (3.25). If $\tau > 1/\gamma^{1+\sigma}, \sigma > 0$, it is easy to show (by Liapunov methods) that $|\eta_{oR}(\tau) - \eta_o^{(r)}(\tau)| \equiv |u(\tau)| < \gamma^{r+1} k(\tau)$ (cf. (3.21)), so that we can replace the right hand side of (3.26) through the rest after an (r,L) truncation of (3.7) with $q = 0$.

In a strictly similar manner, we derive solutions $\eta_{qR}(\tau)$ of (3.6) and analogues. We write then an inner expansion:

$$x_{iR}^{(Q)}(t) = \varepsilon^{1/8} \sum_{q=1}^{Q} \eta_{qR}(\tau)\, \varepsilon^{3q/4} = \varepsilon^{1/8}\, \eta_R^{(Q)}(t) \qquad (3.27)$$

and an approximant to a solution of (1.3) for $t > 0, \Delta < C\Gamma^{1/4}$ as:

$$x_{aR}(t) = \chi_i(t;\varepsilon^\alpha)\, x_{iR}^{(Q)}(t) + \chi_{oR}(t;\varepsilon^\alpha)\, x_o^{(K,L)}(t) \qquad (3.28)$$

where χ_{oR} is analogous to χ_{oL} of (3.8), $\chi_i + \chi_{oR} = 1$ for $0 \leq t \leq \pi/2$. If the power α defining the interval $[a,b]\varepsilon^\alpha$ where the matching of x_{iR}, x_o is performed is such that $\varepsilon^{\alpha-3/8} > 1/\gamma$, the relevant asymptotic expansion of $\varepsilon^{-1/8} x_{iR}^{(Q)}$ for the calculation of $\mathcal{D}(x_{aR})$ is the same as that of $\varepsilon^{-1/8} x_{iL}^{(Q)}$, cf. eqn. (3.7). Thus, $|\mathcal{D}(x_{aR})| \sim |\mathcal{D}(x_{aL})|$, if the truncation numbers Q,K,L are the same. This situation occurs at high damping (near the line $\Delta \sim C\Gamma^{1/4}$). If, however, $\varepsilon^{\alpha-3/8} < 1/\gamma$ as is the case if, e.g. $\Delta \sim c \ln \Gamma$, we must take into account the additional term $O(\gamma^{r+1})$, eqn. (3.26), in the calculation of $\mathcal{D}(x_{aR})$. However, in this situation, $\gamma \sim \varepsilon^q$, for some $q > 0$, so that we can state, in analogy to Lemma 3.3:

<u>Lemma 3.5</u>: Let $P = \min(Q,K,L)$, $S = \min(Q,K,L,r)$. Then:

$$\sup_{0 \leq t \leq \pi/2} |\mathcal{D}(x_{aR})(t)| = O(\varepsilon^{c_1 P(S)}) \qquad (3.29)$$

for some $c_1 > 0$; the estimate in brackets is valid if $\varepsilon^{\alpha-3/8} < 1/\gamma$.

We can now establish the existence of a solution $X_R(t;Q,K,L,r;\varepsilon,\mu)$ of (1.6), as close uniformly as we wish (together with its first two derivatives) on $[0,\pi/2]$ to $x_{aR}(t)$ (provided P (S) are large enough). The procedure is the same as the one for proving the existence of X_L, Theorem 3.1. The solutions $v_{1,2}$ of (3.14) are now chosen so that the corresponding $w_{1,2}$ (eqn. (3.15)) obey the same initial conditions as the WKB functions (3.19) at $t = \pi/2$. Further, the function $\phi(\tau)$, eqn. (3.16), obtained from $\eta_{aR}(\tau)$ has now oscillations of frequency $\sim\tau^{1/3}$ and amplitude $\sim\gamma^{r+1}$ on a time scale $1/\gamma$; if r is sufficiently large, the estimates for (3.19) are nevertheless the same as for x_{aL}.

Due to the damping factor $e^{-\Delta t}$ in $v_{1,2}(t)$, we are constrained to choose the initial conditions for $X_R(t)$ (the same as for $x_{aR}(t)$) at $t = 0$, rather than at $t = \pi/2$, unless the quantity k, eqn. (1.7), is O(1). Indeed, the kernel of (3.13) cannot otherwise be bounded in a useful manner. We can thus only state:

<u>Theorem 3.2</u>: Eqn. (1.6) admits of a solution $X_R(t;Q,K,L,r;\varepsilon,\mu)$ uniformly approximated, together with its first two derivatives, better than $\varepsilon^{c_2 P(S)}$ on $[0,\pi/2]$ by $x_{aR}(t)$, eqn. (3.28) and obeying $X_R(t{=}0) = x_{aR}(t{=}0)$, $dX_R/dt(t{=}0) = dx_{aR}/dt\ (t{=}0)$.

<u>Comment 3.2</u>: Although $x_{aR}(\pi/2) = -x_{aL}(-\pi/2)$, it is not true that $X_R(\pi/2) = -X_L(-\pi/2)$, because of the different choice of initial conditions. The same is true for the derivatives. We can thus only state that, for some $c > 0$:

$$X_R(\pi/2) + X_L(-\pi/2) = A_1(\varepsilon) = O(\varepsilon^{cP(S)}) \neq 0 \tag{3.30}$$

$$\frac{dX_R}{dt}(\pi/2) + \frac{dX_L}{dt}(-\pi/2) = A_2(\varepsilon) = O(\varepsilon^{cP(S)}) \neq 0 \tag{3.31}$$

<u>Comment 3.3</u>: If $k(\varepsilon)$, eqn. (1.17) is O(1), we can make $A_1(\varepsilon) = A_2(\varepsilon) = 0$ (cf. Ref.[15]). Since the quantity P(S) in (3.30),(3.31) is at our disposal, we may nevertheless keep A_1, A_2 nonzero in the following.

<u>Comment 3.4</u>: Notice, at $\tau = 0$, $X_R(0) = \varepsilon^{1/8}[\eta_{oo}(\tau{=}0) + O(\gamma)]$, $dX_R/d\tau\ (\tau{=}0) = \varepsilon^{1/8}[d\eta_{oo}/d\tau\ (\tau{=}0) + O(\gamma)]$. But: $X_L(0) \simeq \varepsilon^{1/8}\eta_{oL}(\tau{=}0;\gamma)$, $dX_L/d\tau\ (\tau{=}0) = \varepsilon^{1/8}[d\eta_{oL}/d\tau\ (\tau{=}0) + O(\varepsilon^{3/4})]$. Now, $\eta_{oL}(\tau{=}0) = \eta_{ooL}(\tau{=}0) + O(\gamma)$, where η_{ooL} is the solution of (3.22) with the boundary condition $\eta_{ooL}(\tau)\sim\tau^{1/3}$ as $\tau\to-\infty$. It is easy to see that $\eta_{ooL}(\tau) = -\eta_{oo}(-\tau)$. Since $\eta_{oo}(\tau{=}0) \neq 0$, $d\eta_{oo}/d\tau\ (\tau{=}0) \neq 0$, it follows that:

$$X_R(\tau{=}0) - X_L(\tau{=}0) \equiv \varepsilon^{1/8}\Delta\eta \neq 0 \tag{3.32}$$

$$\frac{dX_R}{d\tau}(\tau{=}0) - \frac{dX_L}{d\tau}(\tau{=}0) \equiv \varepsilon^{1/8}\Delta'\eta \neq 0 \tag{3.33}$$

if ε is small enough. For small γ, $\Delta\eta = 2\eta_{oo}(0)$, $\Delta'\eta = 2\, d\eta_{oo}/d\tau\,(0)$.

<u>Comment 3.5</u>: Eqn. (1.6) admits also of the following solutions:

$$X_{L1}(t) = -X_L(t-\pi), \quad X_{R1}(t) = -X_R(t-\pi) \tag{3.34}$$

defined for $\pi/2 < t < \pi$, $\pi < t < 3\pi/2$, in turn. Clearly, $X_{L2}(t) = X_L(t-2\pi)$, etc. are also solutions.

With this, given a solution x(t) of (1.6), we can define its successive departures from $X_L(t)$, $X_R(t)$, etc, for $t > -\pi/2$:

$$\begin{aligned} x(t) &= X_L(t) + v_L(t), \quad -\pi/2 < t < 0; = X_R(t) + v_R(t), \quad 0 < t < \pi/2 ; \\ &= X_{L1}(t) + v_{L1}(t), \quad \pi/2 < t < \pi; = X_{R1}(t) + v_{R1}(t), \pi < t < 3\pi/2; \text{etc.} \end{aligned} \tag{3.35}$$

The time 2π Poincare map $\mathbb{P}_o$ associated to (1.6), with initial time $t = -\pi/2$ (mod 2π) is then:

$$\mathbb{P}_o : (v_L(-\pi/2), dv_L/dt(-\pi/2)) \longrightarrow (v_{L2}(3\pi/2), dv_{L2}/dt(3\pi/2) \tag{3.36}$$

A 2π - odd periodic solution of (1.6) gives rise not only to a fixed point of $\mathbb{P}_o$, but also of:

$$\mathbb{P} : (v_L(-\pi/2), dv_L/dt(-\pi/2) \longrightarrow (-v_{L1}(\pi/2), -dv_{L1}/dt(\pi/2)) \tag{3.37}$$

It is easy to verify that the symmetry $x \to -x$, $t \to t + \pi$ of (1.6) implies that $\mathbb{P}_o = \mathbb{P} \circ \mathbb{P} \equiv \mathbb{P}^2$. Now, with (3.30), (3.34),(3.35):

$$v_{L1}(\pi/2) = -X_{L1}(\pi/2) + X_R(\pi/2) + v_R(\pi/2) = v_R(\pi/2) + A_1(\varepsilon) \tag{3.38}$$

and similarly for the derivatives. Thus, the final formal expression for $\mathbb{P}(\varepsilon, \mu)$: $R^2 \to R^2$,which will be made explicit in the next sections is:

$$\mathbb{P} : (v_L(-\pi/2), dv_L/dt(-\pi/2)) \longrightarrow \big(-v_R(\pi/2) - A_1(\varepsilon), -dv_R/dt(\pi/2) - A_2(\varepsilon)\big) \tag{3.39}$$

IV. On the invariant sets of $\mathbb{P}(\varepsilon;\mu)$

The function $v_L(t)$, eqn. (3.35), is a solution of:

$$\varepsilon \ddot{v}_L + 2\mu \dot{v}_L + 3X_L^2 v_L + 3X_L v_L^2 + v_L^3 = 0 \tag{4.1}$$

and v_R obeys a similar equation with index "R". It is convenient to introduce the independent variables, for $t < 0$, $t > 0$, in turn:

$$\theta_L = \varepsilon^{-1/2} \int_t^{-\tau_o \varepsilon^{3/2}} X_L(t') \sqrt{3}\, dt' \quad ; \quad \theta_R = \varepsilon^{-1/2} \int_{\tau_o \varepsilon^{3/2}}^{t} X_R(t') \sqrt{3}\, dt' \tag{4.2}$$

for a τ_o such that $\eta_L(\tau)$, $\eta_R(\tau) \neq 0$ for $|\tau| > \tau_o$. For ease of notation, the index "L" or "R" on θ will be dropped if it is clear whether $t < 0$ or $t > 0$. The aim of this Section is to prove:

Theorem 4.1: For ε small enough, the invariant sets of the Poincaré map $\mathbb{P}(\varepsilon;\mu)$, eqn. (3.39), are contained in a disk:

$$D: \ |v_L|^2 + |dv_L/d\theta_L|^2 < \Lambda_M \, \varepsilon^{3/16+k\pi/2} \tag{4.3}$$

where Λ_M is independent of ε and $k(\varepsilon)$ of (1.17).

A set S is invariant if $\mathbb{P}(S) = S$. Because bifurcations are the main concern, for which (as will turn out) $k(\varepsilon) = O(1)$, we use in the next sections the notation ε^{kt} rather than $\exp(-\frac{\mu}{\varepsilon} t)$. The proof of Theorem 4.1 is achieved in several steps. First, we notice:

Lemma 4.1: All solutions of (4.1) obey eventually (mod 2π):

$$|v_L(-\pi/2)| < K\varepsilon^{k\pi/4} \quad , \quad |\dot{v}_L(-\pi/2)| < K\varepsilon^{-1/2}\,\varepsilon^{k\pi/4} \ . \tag{4.4}$$

This is simply Lemma 2.2: let $C = 3/4$, $A \to 0$ $(\Delta < C\Gamma^{1/4})$, $t_1 = \pi/6$, $t_2 = \pi/2$. It follows that all invariant sets of $\mathbb{P}$ are contained in a disk of radius $\varepsilon^{k\pi/4}$.

Lemma 4.2: Consider a rectangle:

$$D_\alpha: \ |v_L(-\pi/2)| \ , \ |dv_L/d\theta_L(-\pi/2)| \ < \ A\varepsilon^\alpha \tag{4.5}$$

There exist T_{oL}, B, independent of ε, so that all solutions of (4.1) starting in D_α at $t = -\pi/2$ obey at $t = -T_{oL}\varepsilon^{3p}$:

$$|v_L| \ , \ |dv_L/d\theta| < \ B\varepsilon^p \tag{4.6}$$

where $p = \min\left[(k\pi + 2\alpha)/3, \ 1/8\right]$.

Proof: We use the new variable w:

$$v_L \equiv \frac{w}{(-X_L)^{1/2}} \ \varepsilon^{k(t+\pi/2)}\,\varepsilon^\alpha \tag{4.7}$$

which obeys:

$$\frac{d^2w}{d\theta^2} + w(1+g) - w^2 h + \frac{w^3}{3} h^2 = 0 \tag{4.8}$$

where $(\theta = \theta_L)$:

$$h(\theta) = \frac{\varepsilon^{k(t+\pi/2)}}{(-X_L)^{3/2}}\,\varepsilon^\alpha \quad ; \quad g(\theta) = \frac{\varepsilon}{4}\frac{1}{X_L^4}\left(\frac{dX_L}{dt}\right)^2 - \frac{\varepsilon}{6}\frac{1}{X_L^3}\frac{d^2X_L}{dt^2} - k\,\varepsilon\ln^2\frac{1}{\varepsilon} \tag{4.9}$$

Since $g \sim \varepsilon(t^{-8/3} + \ln^2\frac{1}{\varepsilon})$ we notice $g \sim 1$ at $t \sim \varepsilon^{3/8}$, whereas $h \sim 1$ at $t \sim \varepsilon^{k\pi + 2\alpha}$, if $k\pi + 2\alpha \leq 3/8$, but stays otherwise $o(1)$ down to $t = -\tau_1\varepsilon^{3/8}$, $\tau_1 < \tau_o$. Consider the energy associated to (4.8):

$$E(\theta) = \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 + \frac{w^2}{2}(1+g) - \frac{w^3}{3}h + \frac{w^4}{12}h^2 \tag{4.10}$$

for which:

$$\frac{dE}{d\theta} = \frac{w^2}{2}\left(\frac{dg}{d\theta}\right) - \frac{w^3}{3}\frac{dh}{d\theta} + \frac{w^4}{6} h \frac{dh}{d\theta} \tag{4.11}$$

If t is such that $|g| < 1/24$ (i.e. $t < -C\,\varepsilon^{3/8}$), it is true that:

$$|w(\theta)| < \sqrt{12\, E(\theta)} \quad , \ |dw/d\theta| < \sqrt{2\, E(\theta)} \tag{4.12}$$

for $-\pi/2 < t < -C\,\varepsilon^{3/8}$, so that (4.11) implies:

$$\left|\frac{dE}{d\theta}\right| < 6\, E(\theta)\left[\,\left|\frac{dg}{d\theta}\right| + \frac{4}{\sqrt{3}}\left|\frac{dh}{d\theta}\right|\sqrt{E} + 12\, E(\theta)\, h\left|\frac{dh}{d\theta}\right|\,\right] \tag{4.13}$$

At $t = -\pi/2$, $E < E_o(A)$, and we can assume $E_o > 1$. From (4.13) we deduce that, as long as $E(\theta) > 1$:

$$\frac{dE}{d\theta} < 6\, E^2\left[\left|\frac{dg}{d\theta}\right| + 4\left|\frac{dh}{d\theta}\right|\right] \tag{4.14}$$

if ε, T_{oL} are chosen so that $|h(\theta)| < 1/12$, for $t \in [-\pi/2, -T_{oL}\,\varepsilon^{3p}]$. The differential inequality (4.14) can be integrated to yield, for $t > -\pi/2$

$$E(\theta) < \frac{E(\theta_o)}{1 - 6\, E(\theta_o)(\Delta g + 4\Delta h)} \tag{4.15}$$

with Δg, $\Delta h = g(\theta) - g(\theta_o)$, $h(\theta) - h(\theta_o)$ and $\theta_o = \theta_L(-\pi/2)$. We can now choose T_o e.g. so that $\Delta g(T_{oL}\,\varepsilon^{3p})$, $\Delta h(T_{oL}\,\varepsilon^{3p}) < 1/(12\, E_o)$. Then (4.15) implies $E(-T_{oL}\,\varepsilon^{3p}) < 2\, E(-\pi/2)$. Returning to (4.7) and using $(-X_L) \sim |t|^{1/3}$, we obtain the statement of the Lemma.

Notice, we do not yet compare ε^p in (4.6) with ε^α in (4.5). Both situations $\varepsilon^p < \varepsilon^\alpha$ and $\varepsilon^p > \varepsilon^\alpha$ are possible. We evaluate next v_R, $dv_R/d\theta_R$ at $t = T_{oR}\,\varepsilon^{3p}$.

<u>Lemma 4.3</u>: If $|v_L|$, $|dv_L/d\theta_L|$ obey (4.6) at $t = -T_{oL}\,\varepsilon^{3p}$, then:

$$|v_R| \ , |dv_R/d\theta_R| < C\,\varepsilon^p \tag{4.16}$$

at $t = T_{oR}\,\varepsilon^{3p}$, for C independent of ε (but depending on T_{oR}, B).

<u>Proof</u>: In eq. (4.1) for $v_L(t)$ we rescale variables to:

$$t = \varepsilon^{1/2 - p}\sigma \quad , \quad v_L = \varepsilon^p\, V \tag{4.17}$$

so that:

$$\frac{d^2V}{d\sigma^2} + 2\mu\varepsilon^{p-1/2}\frac{dV}{d\sigma} + 3\,(X_L^2\,\varepsilon^{-2p})\, V + 3(X_L\,\varepsilon^{-p})\, V^2 + V^3 = 0 \tag{4.18}$$

It is true that, for $t \in [-T_{oL}\,\varepsilon^{3p}, 0]$, $|X_L\,\varepsilon^{-p}| < T_{oL}^{1/3}$, independently of ε. The energy associated to (4.18) is:

$$E = \frac{1}{2}\left(\frac{dV}{d\sigma}\right)^2 + \frac{3}{2}\,(X_L\,\varepsilon^{-p})^2\, V^2 + (X_L\,\varepsilon^{-p})\, V^3 + \frac{V^4}{4} \tag{4.19}$$

At $\sigma = -(T_{oL}\,\varepsilon^{3p})\,\varepsilon^{p-1/2} \equiv \sigma_a$, one verifies $E(\sigma) < B_1$, independently of ε, if (4.6) holds. Further, for $t \in [-T_{oL}\,\varepsilon^{3p}, 0]$:

$$|V| < E^{1/4}/2 \tag{4.20}$$

as one verifies from (4.19). We derive then from (4.18), (4.19) the differential inequality:

$$\left|\frac{dE}{d\sigma}\right| < \frac{E^{3/4}}{8}\left[\left|\frac{d}{d\sigma}(X_L\,\varepsilon^{-p})\right| + \frac{3}{2}\left|\frac{d}{d\sigma}(X_L\,\varepsilon^{-p})^2\right|\right] \tag{4.21}$$

Integration of (4.21) yields:

$$E(\sigma=0)^{1/4} < E(\sigma_a)^{1/4} + \frac{1}{32}\left[\frac{3}{2}T_{oL}^{2/3} + T_{oL}^{1/3}\right] \equiv C_1 \tag{4.22}$$

with C_1 independent of ε. With (4.19), (4.20) this means

$$|v_L(t=0)| < C_2\,\varepsilon^{p} \quad , \quad |dv_L/d\sigma\,(t=0)| < C_3\,\varepsilon^{p} \tag{4.23}$$

for some C_2, $C_3 > 0$. Using the discontinuity rules (3.32-33), we transfer the information (4.23) on v_L to similar inequalities on v_R:

$$|v_R(0)| < |v_L(0)| + |(X_R - X_L)(t=0)| < C_2\,\varepsilon^{p} + \varepsilon^{1/8}\Delta\gamma < C_4\,\varepsilon^{p} \tag{4.24}$$

$$\left|\frac{dv_R}{d\sigma}(0)\right| < \left|\frac{dv_L}{d\sigma}(0)\right| + \left|\frac{d}{d\sigma}(X_R - X_L)(0)\right| < C_3\,\varepsilon^{p} + \varepsilon^{1/4-p}\Delta'\gamma < C_5\,\varepsilon^{p} \tag{4.25}$$

where we have used the fact that $p \leqslant 1/8$.

We can now repeat the reasoning above for the analogue of (4.1) with index "R" and conclude, similarly to (4.22), that:

$$E(\sigma = T_{oR}\,\varepsilon^{4p-1/2})^{1/4} < C_o + D\,T_{oR}^{2/3} \tag{4.26}$$

with constants C_o, D, T_{oR} independent of ε. Using (4.20), (4.19) and returning to the variable θ_R, we obtain (4.16). This ends the proof.

<u>Lemma 4.4</u> : If v_R, $dv_R/d\theta_R$ obey (4.16) at $t = T_{oR}\,\varepsilon^{3p}$, then:

$$|v_R(t=\pi/2)| \;,\; |dv_R/d\,\theta_R(t=\pi/2)| < M\,\varepsilon^{(3p+k\pi)/2} \tag{4.27}$$

for a constant M independent of ε.

<u>Comment 4.1</u>: If $p = (k\pi + 2\alpha)/3 < 1/8$, then the bound in (4.27) is strictly smaller than the original size ε^{α}, eqn. (4.5), of the set of initial conditions by a factor $\varepsilon^{k\pi}$. Thus, at every half period α increases by $k\pi$ until we reach the situation $p = 1/8$. In this latter case, (4.27) means $|v_R|, |dv_R/d\theta| < M\,\varepsilon^{3/16+k\pi/2}$. Choosing $\alpha = 3/16 + k\pi/2$ in (4.5), we obtain again $p = 1/8$. Thus Lemma 4.4 implies indeed that all invariant sets of $\mathcal{P}$ are contained in a disk of size

$\Lambda_M \varepsilon^{3/16 + k\pi/2}$ in the $(v_L, dv_L/d\theta_L)$ plane, for some $\Lambda_M > 0$. This is precisely the statement of Theorem 4.1.

Comment 4.2: The statement of Theorem 4.1 (and of Lemma 4.4) is of interest only if $k = O(1)$. If $k \to \infty$ as $\varepsilon \to 0$, it gives qualitatively the same bounds as Lemma 2.2.

To prove Lemma 4.4, we write, similarly to (4.7), ($\theta_R \equiv \theta$):

$$v_R(\theta) = \frac{w(\theta)}{X_R^{1/2}} \varepsilon^{kt} \varepsilon^{3p/2} \tag{4.28}$$

We verify that, at $t = T_{oR}\,\varepsilon^{3p}$:

$$|w(\theta)|,\ |dw/d\theta| < C_1(T_{oR}) \tag{4.29}$$

where the numbering of the constants C begins anew and $C_1(T_{oR}) \sim T_{oR}^{2/3}$, but is independent of ε. The function $w(\theta)$ obeys:

$$\frac{d^2w}{d\theta^2} + w(1+g) + w^2 h_1 + \frac{w^3}{3} h_1^2 = 0 \tag{4.30}$$

with:

$$h_1(\theta) = \frac{\varepsilon^{kt + 3p/2}}{X_R^{3/2}} \tag{4.31}$$

and $g(\theta)$ of (4.9), now with index "R". The proof of Lemma 4.4 is finished if we verify that the energy associated to (4.30) is bounded by a constant even at $\theta(\pi/2)$.

We cannot, however, make direct use to this end of the argument leading to Lemma 4.2. The reason is, we cannot make sure we can find a T_{oR} so that the denominator in (4.15) is bounded from below by a positive constant at $\theta_a = \theta(T_{oR}\,\varepsilon^{3p})$. Indeed, if, e.g. $p < 1/8$, we can only state: $E(\theta_a)(|\Delta g| + 4|\Delta h|) < (C_2 + C_3 T_{oR}^{8/3})\, T_{oR}^{-1/2}$ and we have no reason why this bound should be less than 1/6, for some T_{oR}.

Now, if δ is such that $0 < \delta < 3p$, it is true that, at $t_1 = \varepsilon^{3p - \delta}$, $\theta_1 = \theta(t_1)$, $h_1(t_1) = O(\varepsilon^{\delta/2})$. It follows that the quantities $\Delta h_1 = h_1(\theta_1) - h_1(\theta(\pi/2))$, $\Delta g = g(\theta_1) - g(\theta(\pi/2))$, which appear in (4.15) may be made as small as one wishes, by letting ε be small enough. We can thus state:

Lemma 4.5: Assume:

$$|w(\theta_1)|,\ |dw/d\theta(\theta_1)| < C_4 \tag{4.32}$$

with $\theta_1 = \theta(\varepsilon^{3p - \delta})$ and C_4 independent of ε. Then, if ε is sufficiently small:

$$|w(\theta(\pi/2)|, \quad |dw/d\theta(\theta(\pi/2))| < C_5 \tag{4.33}$$

for a constant C_5 independent of ε.

The proof is identical to that of Lemma 4.2.

The proof of Lemma 4.4 is thus finished if we justify:

Lemma 4.6: The solutions $w(\theta)$ of (4.30) map the rectangle (4.29) at $\theta_o = \theta(T_{oR}\,\varepsilon^{3p})$ into the interior of a rectangle (4.32) at $\theta_1 = \theta(\varepsilon^{3p-\delta})$, with C_2 independent of ε, if ε is sufficiently small.

The proof is displayed in Appendix C.

With Comment 4.1, the proof of Theorem 4.1 is completed.

V. The left hand side Poincaré mapping $\mathbb{P}_L$.

In this and the next sections, we derive controlled approximations to the half period Poincaré map $\mathbb{P}$, eqn. (3.39). We discuss first the quarter period map $\mathbb{P}_L$:

$$\mathbb{P}_L : (v_L(-\pi/2),\ dv_L/d\theta(-\pi/2)) \to \varepsilon^{1/8}(v_L(0),\ dv_L/d\tau(0)) \tag{5.1}$$

with $\theta = \theta_L$ of (4.2) and τ of (1.15). Clearly, in view of Theorem 4.1, for the discussion of uniqueness or of bifurcations, it is enough to restrict the domain of $\mathbb{P}_L$ to the disk D, eqn. (4.3). We parametrize then:

$$v_L(-\pi/2) = \varepsilon^{3/16 + k\pi/2}\,\Lambda \cos\Psi_o \tag{5.2}$$

$$\frac{dv_L}{d\theta}(-\pi/2) = -\varepsilon^{3/16 + k\pi/2}\,\Lambda \sin\Psi_o \tag{5.3}$$

The function $v_L(t)$ is a solution of (4.1), $-\pi/2 < t < 0$. Most of this Section is devoted to the proof of the following:

Theorem 5.1: Consider the linear equation:

$$\varepsilon \frac{d^2\tilde{v}_L}{dt^2} + 2\mu \frac{d\tilde{v}_L}{dt} + 3X_L^2\,\tilde{v}_L = 0 \tag{5.4}$$

with the initial conditions at $t = -\pi/2$

$$\tilde{v}_L(-\pi/2) = \Lambda \cos(\Psi_o + \phi_{so}(\varepsilon;\Lambda))\,\varepsilon^{3/16 + k\pi/2} \tag{5.5}$$

$$\frac{d\tilde{v}_L}{d\theta}(-\pi/2) = -\Lambda \sin(\Psi_o + \phi_{so}(\varepsilon;\Lambda))\,\varepsilon^{3/16 + k\pi/2} \tag{5.6}$$

where:

$$\phi_{so}(\varepsilon) = -\frac{7\sqrt{3}}{24}\int_{-\pi/2}^{-\tau_0\varepsilon^{3/8}} \Lambda^2\, \varepsilon^{-1/8}\, \varepsilon^{k(t+\pi/2)}\, \frac{1}{X_L^2}\, dt \qquad (5.7)$$

and τ_o of (4.2). Let further v_L be the solution of (4.1) with initial conditions (5.2-3) at $t = -\pi/2$ and:

$$u_L = \varepsilon^{-1/8}\, v_L \quad , \quad \tilde{u}_L = \varepsilon^{-1/8}\, \tilde{v}_L \quad . \qquad (5.8)$$

Then, at $t = 0$:

$$u_L,\ \tilde{u}_L\ ,\ \frac{du_L}{d\tau}\ ,\ \frac{d\tilde{u}_L}{d\tau} = 0(\varepsilon^{k\pi}) \qquad (5.9)$$

and:

$$\varepsilon^{-k\pi}(u_L - \tilde{u}_L)\ (t = 0) = 0(\varepsilon^{k\pi/2}) \qquad (5.10)$$

$$\varepsilon^{-k\pi}\left(\frac{du_L}{d\tau} - \frac{d\tilde{u}_L}{d\tau}\right)(t = 0) = 0(\varepsilon^{k\pi/2}) \qquad (5.11)$$

Comment 5.1: This theorem states that we can compute $\mathbb{P}_L$ for v_L, $dv_L/d\theta$ $(-\pi/2)$ inside D, eqn. (4.3), simply by means of the linearized equation (5.4), with a precision increasing indefinitely as $\varepsilon \to 0$.

Comment 5.2: It is not true that the solution of (5.4) approximates the relevant one of (4.1) on all of $(-\pi/2, 0)$; this happens only on some interval near $\tau = 0$.

The proof of Theorem 5.1 proceeds via a number of Lemmas, which we display below. We use the variable w of (4.7), with $\alpha = 3/16 + k\pi/2$ (cf. (4.9)):

$$v_L = \frac{w}{(-X_L)^{1/2}}\, \varepsilon^{k(t+\pi/2)}\, \varepsilon^{3/16 + k\pi/2} \equiv (-X_L)\ w\ h(\theta) \qquad (5.12)$$

According to Lemma 4.2, $M < \infty$ exists so that:

$$|w(\theta)|,\ |dw/d\theta| < M, \quad t \in [-\pi/2, -T_{OL}\,\varepsilon^{3/8}] \equiv I_L(\tau;\varepsilon) \qquad (5.13)$$

The fact that $h(\theta)$ in eqn. (4.8) is a small quantity invites the use of averaging methods (Refs.[22,23]) to estimate $w(\theta)$, $dw/d\theta$ on $I_L(\tau;\varepsilon)$:

Lemma 5.1: Let:

$$w(\theta) = R(\theta)\cos(\theta - \theta_o + \varphi(\theta)) \qquad (5.14)$$

$$\frac{dw}{d\theta}(\theta) = -R(\theta)\sin(\theta - \theta_o + \varphi(\theta)) \qquad (5.15)$$

with $\theta_o = \theta_L(-\pi/2)$. The following estimates hold:

$$R(\theta) = \Lambda_{5o} + 0\left[h(\theta) + g(\theta) + \int_{\theta_o}^{\theta}(h^4 + gh^2)\, d\theta'\right] \qquad (5.16)$$

$$\varphi(\theta) = \phi_{sL}(\theta;\varepsilon) + \int_{\theta_o}^{\theta}\frac{g(\theta')}{2}\, d\theta' + \Psi_{4o} + 0\left[h(\theta) + g(\theta) + \int_{\theta_o}^{\theta}(h^3 + gh^2)\, d\theta'\right] \qquad (5.17)$$

where $\Lambda_{5o}(\Lambda, \psi_o, \varepsilon)$, $\Psi_{4o}(\Lambda, \psi_o, \varepsilon)$ approach Λ, ψ_o, eqns. (5.2),(5.3) in turn, as $\varepsilon \to 0$ and:

$$\Phi_{sL}(\theta;\varepsilon) = -\frac{7}{24}\int_{\theta_0}^{\theta}\Lambda_{50}^2 \quad h^2(\theta';\varepsilon)d\theta' \tag{5.18}$$

The estimates under the O(·) sign in (5.16-17) depend on M, eqn. (5.13).

Proof: In eqn. (4.8) we change to polar coordinates $R(\theta)$, $\varphi(\theta)$ as in (5.14-15). Then:

$$\frac{dR}{d\theta} = R \sin z \quad \cos z \left[g - hR\cos z + \frac{h^2R^2}{3}\cos^2 z\right] \tag{5.19}$$

$$\frac{d\varphi}{d\theta} = \cos^2 z\left[g - hR\cos z + \frac{h^2R^2}{3}\cos^2 z\right] \tag{5.20}$$

with

$$z = \theta - \theta_o + \varphi \tag{5.21}$$

Consider first eqn. (5.19). We perform on $R(\theta)$ a sequence of transformations of the averaging type to eliminate successively the terms in h, h^2, h^3, g, hg, which have zero average with respect to z; for definiteness, they read:

$$R_1 = R - \frac{hR^2}{3}\cos^3 z \quad ; \qquad R_2 = R_1 + \frac{R^3h^2}{3}\left(\frac{1}{16}\cos 2z - \frac{1}{48}\cos^3 2z\right) \quad ; \tag{5.22}$$

$$R_3 = R_2 + \frac{h^3R^4}{12}\cos^3 z\left(\frac{1}{3} - \frac{1}{2}\cos^2 z + \frac{19}{21}\cos^4 z - \frac{1}{2}\cos^6 z\right) \quad ;$$

$$R_4 = R_3 + \frac{gR}{4}\cos 2z \quad ; \quad R_5 = R_4 + ghR^2\left(\frac{7}{30}\cos^5 z - \frac{1}{12}\cos^3 z\right).$$

Notice, R_i, $i=1,2,\ldots$ is defined in terms of R_{i-1} and the original variable R; at each stage, one may imagine that we have solved $R = R(R_{i-1}, z; h, g)$. We also use $dz/d\theta = 1 + d\varphi/d\theta$ and (5.20). The important occurrence is that, at every step, the coefficients of h, h^2, h^3, g, hg are, in turn, trigonometric polynomials of z with zero average. With this, the equation satisfied by R_5 is:

$$\frac{dR_5}{d\theta} = O\left(h^4 + gh^2 + \left|\frac{dg}{d\theta}\right|\right) \tag{5.23}$$

An absolute bound on the right hand side of (5.23) is possible because $R(\theta)$ is bounded on $I_L(\tau;\varepsilon)$ by M, eqn. (5.13) and the coefficients of the various powers of $R(\theta)$ are trigonometric polynomials of z. Eqn. (5.23) means:

$$R_5 - \Lambda_{50} = O\left[\int_{\theta_0}^{\theta} h^4 d\theta' + \int_{\theta_0}^{\theta} gh^2 d\theta' + g(\theta) - g(\theta_o)\right] \tag{5.24}$$

where:

$$\Lambda_{50} = R_5(\Lambda, \theta_o, \Psi_o, \varepsilon) \tag{5.25}$$

is obtained by letting $z = \Psi_o$, $R = \Lambda$, $\theta = \theta_o$ in the sequence (5.22). Clearly, $\Lambda_{50} \to \Lambda$ as $\varepsilon \to 0$. The sequence (5.22) can be inverted for ε small enough, T_{oR} large enough and we obtain (5.16). In (5.16), the terms of $O(h + g)$ are, in principle, known functions of z. However, only estimates are available for the other terms.

We transform next eqn. (5.20), using (5.19) to eliminate successively terms in h, h^2, g, hg:

$$\varphi_1 = \varphi + hR \sin z \left(1 - \frac{\sin^2 z}{3} \right) ; \ \varphi_2 = \varphi_1 + h^2R^2\left(\frac{5}{24} \sin 2z + \frac{1}{96} \sin 4z + \frac{1}{288}\sin 6z \right); \varphi_3 = \varphi_2 - \frac{g}{4} \sin 2z; \varphi_4 = \varphi_3 + hgR \sin z\left(\frac{3}{2} + \frac{5}{6} \sin^2 z + \frac{1}{3} \sin^4 z \right) \tag{5.26}$$

In contrast to (5.23), the averaged equation for φ_4 does contain terms in g and h^2:

$$\frac{d\varphi_4}{d\theta} = \frac{g}{2} - \frac{7}{24} h^2R^2 + O(h^2 g + h^3 + | dg/d\theta |) \tag{5.27}$$

Integrating (5.27), moving back to φ by inverting the steps in (5.26) and using (5.16) for $R(\theta)$, we obtain (5.17). This ends the proof of Lemma 5.1.

<u>Comment 5.3</u>: We may replace Λ_{50}, Ψ_{40} in (5.16), (5.17) by (Λ, Ψ_o) at the price of adding terms (known in principle) of $O(h_o + g_o)$ under the $O(\cdot)$ sign; $h_o = h(\theta_o), g_o = g(\theta_o)$.

<u>Comment 5.4</u>: It turns out that the terms of $O(h^3)$ under the $O(\cdot)$ sign in (5.27) have zero average with respect to z. Also, if in (5.27) we replace

$$R = R_5 + \frac{hR_5^2}{3} \cos^3 z + O(h^2 + g) \tag{5.28}$$

we obtain additional terms of $O(h^3)$ with zero average. We can perform thus a further transformation $\varphi_5 = \varphi_4 + O(h^3)$ to remove them; the resulting equation for φ_5 reads then:

$$\frac{d\varphi_5}{d\theta} = \frac{g}{2} - \frac{7}{24} h^2 R_5^2 + O(h^4 + h^2 g + |\frac{dg}{d\theta}|) \quad . \tag{5.29}$$

The similarity of the $O(\cdot)$ terms with those of (5.23) is of use in Theorem 3.2 below.

We state next:

<u>Lemma 5.2</u>: The solution of the linear equation (5.4) with the boundary conditions (5.5-6) is given on $I_L(\tau; \varepsilon)$ by:

$$\tilde{v}_L = \frac{\tilde{w}(\theta)}{(-X_L)^{1/2}} \varepsilon^{k(t+\pi/2)} \varepsilon^{3/16+k\pi/2} \equiv (-X_L) \tilde{w}h(\theta) \tag{5.30}$$

where:

$$\tilde{w}(\theta) = \tilde{R}(\theta) \cos(\theta - \theta_o + \tilde{\varphi}) \tag{5.31}$$

$$\frac{d\tilde{w}}{d\theta}(\theta) = -\tilde{R}(\theta)\sin(\theta - \theta_o + \tilde{\varphi}) \tag{5.32}$$

and(cf.eqn.(5.7)):

$$\tilde{R}(\theta) = \Lambda + O(g + g_o) \tag{5.33}$$

$$\tilde{\varphi}(\theta) = \tilde{\varphi}_o + \int^{\theta} \frac{g(\theta')}{2}\, d\theta' + O(g + g_o) \tag{5.34}$$

$$\tilde{\varphi}_o = \psi_o + \Phi_{so}(\varepsilon) \tag{5.35}$$

The proof is done in the same way as for Lemma 5.1, with h = 0 (cf.Comment 5.3). Eqns.(5.30-35) are a slight improvement over the WKB approximation.

From Lemmas 5.1, 5.2, we conclude that, if $t \in I_L(\tau;\varepsilon)$:

$$|w(\theta) - \tilde{w}(\theta)|\ ,\ \left|\frac{dw}{d\theta} - \frac{d\tilde{w}}{d\theta}\right| = O\left[h(\theta) + g(\theta) + h_o + g_o + \int_{\theta_o}^{\theta}(h^3 + gh^2)\,d\theta' + \int_{\theta}^{\theta(-\tau_o\varepsilon^{3/8})} h^2\, d\theta'\right] \tag{5.36}$$

Notice, the last term under the O($\cdot$) sign may be divergent as $\varepsilon \to 0$, if k is small enough, for finite values of $t(\theta)$. However,

<u>Lemma 5.3</u>: At $t \simeq \varepsilon^{3/8-\delta}$, $\delta = 3k\pi/16$

$$|u_L(\tau) - \tilde{u}_L(\tau)| = O(\varepsilon^{3k\pi/2 + k\pi/32}) \tag{5.37}$$

$$\left|\frac{du_L}{d\tau}(\tau) - \frac{d\tilde{u}_L}{d\tau}(\tau)\right| = O(\varepsilon^{3k\pi/2 - k\pi/32}) \tag{5.38}$$

with u_L, $\tilde{u}_L$ of (5.8).

This follows by estimating directly the terms under the O($\cdot$) sign in (5.36) using the formulae (5.12), (5.30) and the estimate $X_L(t) \sim \varepsilon^{1/8 - \delta/3}$. With $\delta \leq 3k\pi/16$, the term in $g(\theta)$ is dominant in (5.36). Clearly, there is arbitrariness in the choice of δ .

Further, from (5.12), (5.30) and Lemmas 5.1, 5.2, we see that, at $\tau = -T_{OR}$:

$$u_L\ ,\ \tilde{u}_L,\ \frac{du_L}{d\tau},\ \frac{d\tilde{u}_L}{d\tau} = O(\varepsilon^{k\pi}) \tag{5.39}$$

with τ of (1.15). We introduce thus:

$$U_L\ ,\ \tilde{U}_L = \varepsilon^{-k\pi}(u_L, \tilde{u}_L) \tag{5.40}$$

In the following, we estimate the differences $U_L(\tau) - \tilde{U}_L(\tau)$, $dU_L/d\tau - d\tilde{U}_L/d\tau$ on the τ-interval $[-\varepsilon^{-\delta}, 0]$, by a direct comparison of (5.4) and (4.1) on this interval, using the bounds (5.37-38). The equation satisfied by U_L is:

$$\frac{d^2U_L}{d\tau^2} + 2\gamma\frac{dU_L}{d\tau} + 3\eta_L^2 U_L + 3\eta_L\varepsilon^{k\pi}U_L^2 + \varepsilon^{2k\pi}U_L^3 = 0 \tag{5.41}$$

with γ of (1.15), $\eta_L = X_L/\varepsilon^{1/8}$, whereas $\tilde{U}_L$ is a solution of the linear part of

(5.41) (i.e. of (5.4)). For the comparison, we rewrite (5.41) as an integral equation, using initial conditions at $\tau = -\varepsilon^{-\delta}$ and two linearly independent solutions of (5.4), $v_1(t;\varepsilon)$, $v_2(t;\varepsilon)$. These latter are chosen in the same manner as the two solutions of eqn. (3.14), discussed in Sect. III, with the only change $x_{aL} \to X_L$. We preserve the notation w_1, w_2 of (3.17)-(3.19). Lemma 3.4 is clearly valid. We denote:

$$V_{1,2}(\tau;\varepsilon) \equiv \varepsilon^{kt} w_{1,2}(\tau;\varepsilon) \equiv v_{1,2}(t;\varepsilon)\,\varepsilon^{-k\pi/2} \tag{5.42}$$

For $|t|<\varepsilon^{3/8-\delta}$, with δ of Lemma 5.3, $\varepsilon^{kt} \simeq 1$ and, using (3.20):

$$W(V_1,V_2) = \exp(-2\gamma\tau) \simeq 1 + O(\varepsilon^{3/8-\delta_1}), \qquad \delta_1 > \delta \tag{5.43}$$

Further, from (3.18), (3.17), for $|t|<\varepsilon^{3/8-\delta}$

$$|V_{1,2}(\tau;\varepsilon)| < \min\left[C|\tau|^{-1/6}, D\right], \; |dV_{1,2}/d\tau(\tau;\varepsilon)| < E|\tau|^{1/6} + F \tag{5.44}$$

with constants C, D, E, F independent of ε .

With this, the solution of (5.41) with initial conditions at $\tau = -\varepsilon^{-\delta}$ is the solution of:

$$U_L(\tau) = A\,V_1(\tau) + BV_2(\tau) - \int_{-\varepsilon^{-\delta}}^{\tau}\big(V_1(\tau')\,V_2(\tau) - V_1(\tau)V_2(\tau')\big)\varepsilon^{k\pi}\big(U_L^2 + \varepsilon^{k\pi}\,U_L^3\big)d\tau' \tag{5.45}$$

The solution $\tilde{U}_L(\tau)$ of (5.4), defined in (5.40) may be expressed in terms of $V_{1,2}(\tau;\varepsilon)$ as: (cf. Lemma 5.2)

$$\tilde{U}_L(\tau) = 3^{1/4}\Lambda\left[V_1(\tau)\cos(-\theta_0+\tilde{\varphi}_0) - V_2(\tau)\sin(-\theta_0+\tilde{\varphi}_0)\right] \equiv \check{A}\,V_1(\tau)+\check{B}\,V_2(\tau) \tag{5.46}$$

Now, concerning (5.45), we may state:

<u>Lemma 5.4</u>: The solution of (5.45) is bounded on $[-\varepsilon^{-\delta},0]$, independently of ε .

<u>Proof</u>: For $\tau\in[-\varepsilon^{-\delta},-T_{oR}]$, this follows directly from the estimates (5.16), (5.17). For $|\tau|< T_{oR}$, we write the integral equation with initial conditions at $-T_{oR}$. Letting:

$$r(\tau) = U_L(\tau) - A_1V_1(\tau) - B_1V_2(\tau) \tag{5.47}$$

with A_1, B_1 chosen to match the initial conditions , we verify in a standard manner that the integral equation has a solution $r_o(\tau)$ in a ball of radius $\varepsilon^{k\pi}$, in the space of functions continuous on $[-T_{oR},0]$. The solution is unique because of the local Lipschitz continuity of the integrand with respect to r.

Finally, we can compare directly (5.45) with (5.46). We have:

Lemma 5.5:

$$|U_L(0) - \tilde{U}_L(0)| \quad , \left| \frac{dU_L}{d\tau}(0) - \frac{d\tilde{U}_L}{d\tau}(0) \right| \quad = O(\varepsilon^{k\pi/2}) \tag{5.48}$$

Clearly, the proof of Lemma 5.5 concludes the proof of Theorem 5.1.

Proof: Expressing A, B in (5.45) in terms of $U_L(-\varepsilon^{-\delta})$, $dU_L/d\tau\,(-\varepsilon^{-\delta})$ and similarly $\tilde{A}$, $\tilde{B}$ in (5.46), we obtain, using (5.44), (5.37-38), (5.40):

$$|A - \tilde{A}| < |dV_2/d\tau\,(-\varepsilon^{-\delta})| \; |(U_L - \tilde{U}_L)(-\varepsilon^{-\delta})| + |V_2(-\varepsilon^{-\delta})| \left| \left(dU_L/d\tau - d\tilde{U}_L/d\tau\right)(-\varepsilon^{-\delta}) \right|$$

$$= O(\varepsilon^{k\pi/2}) \tag{5.49}$$

and, analogously, $|B - \tilde{B}| = O(\varepsilon^{k\pi/2})$. Then, using $|U_L(\tau)| < M_1$ (Lemma 5.4), for $\tau \in [-\varepsilon^{-\delta}, 0]$ and (5.49):

$$|U_L(0) - \tilde{U}_L(0)| < \text{const}\; \varepsilon^{k\pi/2} + \int_{-\varepsilon^{-\delta}}^{0} (G|\tau'|^{1/6} + H)\,\varepsilon^{k\pi} M_1^2 \, d\tau' = O(\varepsilon^{k\pi/2}) \tag{5.50}$$

with G,H constants independent of ε. Differentiation of (5.45), (5.46) and subtraction leads to the second estimate in (5.48). This ends the proof.

To summarize, according to Theorem 5.1 and using (5.46), the mapping $\mathbb{P}_L$ is given by:

$$u_L(\varepsilon;0) = 3^{1/4}\Lambda\varepsilon^{k\pi}\left[V_1(0;\varepsilon)\cos(-\theta_o+\tilde{\varphi}_o) - V_2(0;\varepsilon)\,\sin(-\theta_o+\tilde{\varphi}_o)\right] + O(\varepsilon^{3k\pi/2}) \tag{5.51a}$$

$$\frac{du_L}{d\tau}(\varepsilon;0) = 3^{1/4}\Lambda\,\varepsilon^{k\pi}\left[\frac{dV_1}{d\tau}(0;\varepsilon)\cos(-\theta_o+\tilde{\varphi}_o) - \frac{dV_2}{d\tau}(0;\varepsilon)\,\sin(-\theta_o+\tilde{\varphi}_o)\right] + O(\varepsilon^{3k\pi/2}) \tag{5.51b}$$

with $\tilde{\varphi}_o$ of (5.35). With Lemma 3.4, we can even write $V_i(0;0)$ instead of $V_i(\varepsilon;0)$ and, in view of (B.7), the estimates in (5.51) stay unchanged, for small enough k.

Comment 5.5: The quantity $\Phi_{so}(\varepsilon)$, eqn. (5.7), behaves for small ε like:

$$\phi_{so}(\varepsilon) \sim \frac{\varepsilon^{k\pi}}{\gamma^{1/3}} \; \frac{1}{(k\,\ln\frac{1}{\varepsilon})^{2/3}} \tag{5.52}$$

With a view to the next sections, we introduce the quantity:

$$\beta_o \equiv \frac{\varepsilon^{k\pi}}{\gamma^{1/3}} \tag{5.53}$$

If $k(\varepsilon) \geq (8\pi)^{-1}$, $\beta_o = O(1)$ as $\varepsilon \to 0$, and β_o increases as the damping decreases. It will turn out that bifurcations occur if $\beta_o \simeq$ const as $\varepsilon \to 0$. For this reason, it is convenient to take β_o as the second parameter in $\mathbb{P}(\varepsilon;\mu)$, rather than μ. For small ε, one verifies, using (5.53) that:

$$\left(\frac{\partial h}{\partial \beta_o}\right)_{\varepsilon} \simeq \frac{(t+\pi)\,h}{\beta_o} \tag{5.54}$$

Let $U_{Lo} \equiv U_L(\tau = 0)$, $U'_{Lo} = dU_L/d\tau\,(\tau = 0)$ with U_L of (5.40). Clearly, U_{Lo} =

$U_{Lo}(\beta_o;\varepsilon;\Lambda;\Psi_o)$, $U'_{Lo} = U'_{Lo}(\beta_o;\varepsilon;\Lambda;\Psi_o)$. Concerning this dependence, we may state:

Theorem 5.2: If $0<\beta_o<\beta_M$, the functions U_{Lo}, U'_{Lo} have any number of derivatives with respect to Λ, Ψ_o, which are bounded as functions of ε, as $\varepsilon \to 0$. If, in addition, $\beta_o > \beta_m > 0$, U_{Lo}, U'_{Lo} are differentiable with respect to β_o and the derivative is bounded as $\varepsilon \to 0$.

The proof is given in Appendix D.

Comment 5.6: We write in (5.51), with obvious notation:

$$\mathbb{P}_L(\Lambda;\Psi_o) = \varepsilon^{k\pi}\mathbb{P}_L^o + \varepsilon^{3k\pi/2}\mathbb{P}_L^1 \equiv \varepsilon^{k\pi}\tilde{\mathbb{P}}_L(\Lambda;\Psi_o;\varepsilon;\beta) \tag{5.55}$$

From the proof of Theorem 5.2, it follows that the derivatives of $\tilde{\mathbb{P}}_L$ are approximated to $O(\varepsilon^q)$ by those of $\mathbb{P}_L^o$, as $\varepsilon \to 0$, $q>0$.

Comment 5.7: All derivatives of $\mathbb{P}_L$ with respect to Λ, Ψ_o, β_o are continuous as $\varepsilon \to 0$ and are $O(\varepsilon^{k\pi})$.

VI. The right hand side Poincaré mapping $\mathbb{P}_R$ and the complete mapping $\mathbb{P}$.

From the discontinuity formulae (3.32-33), we obtain:

$$u_R(0) = u_L(\tau = 0) - \Delta\gamma \quad , \quad du_R/d\tau\,(0) = du_L/d\tau\,(0) - \Delta'\gamma \tag{6.1}$$

We discuss next the mapping:

$$\mathbb{P}_R : (u_R(0),\, du_R/d\tau\,(0)) \longrightarrow (v_R(\pi/2),\, dv_R/d\tau\,(\pi/2)) \tag{6.2}$$

restricted to a disk of radius $O(\varepsilon^{k\pi})$ around $(-\Delta\gamma, -\Delta'\gamma)$ (cf. (5.51)). To this end, we write, for $\tau > T_{oR}$

$$v_R(\theta) = \frac{w}{X_R^{1/2}}\,\varepsilon^{kt + 3/16} \tag{6.3}$$

(now $\theta = \theta_R$, eqn. (4.2)), so that:

$$-\frac{d^2w}{d\theta^2} + w(1+g) + w^2\,k(\theta) + w^3\,\frac{k^2(\theta)}{3} = 0 \tag{6.4}$$

where:

$$k(\theta) = \frac{\varepsilon^{kt + 3/16}}{X_R^{3/2}} \tag{6.5}$$

The mapping $\mathbb{P}_R$ involves solutions of (6.4) with initial conditions $w, dw/d\theta = O(1)$ at $\theta(T_{oR}\varepsilon^{3/2}) \equiv \theta_a$. By Lemmas 4.5, 4.6, these solutions are bounded on $[\theta_a, \theta_o]$, uniformly with respect to ε.

Lemma 6.1: The solution $w(\theta)$ of (6.4) is given on $[\theta_a, \theta_o]$ by:

$$w(\theta) = R(\theta)\cos(\theta + \varphi(\theta)) \tag{6.6}$$

$$\frac{dw}{d\theta}(\theta) = -R(\theta)\sin(\theta+\varphi(\theta)) \tag{6.7}$$

with: $(\theta_o = \theta_R(\pi/2))$

$$R(\theta) = R_o + O\left[k(\theta) + g(\theta) + \int_\theta^{\theta_o}(k^4 + gk^2)\, d\theta'\right] \tag{6.8}$$

$$\varphi(\theta) = \varphi_o + \varphi_{sR}(\theta) - \int_\theta^{\theta_o}\frac{g(\theta')}{2}\, d\theta' + O\left[k(\theta) + g(\theta) + \int_\theta^{\theta_o}(k^3+gk^2)d\theta'\right] \tag{6.9}$$

and:

$$R_o \equiv R(\theta_o),\ \varphi_o = \varphi(\theta_o),\ \bar{\varphi}_o = \varphi_o - \varphi_{sR}(\theta_o) \tag{6.10}$$

$$\varphi_{sR}(\theta) = -\frac{7}{24}\int_{\theta_a}^{\theta} k^2\, R^2(\theta';R_o,\varphi_o)\, d\theta' \tag{6.11}$$

In (6.8),(6.9), the terms under the $O(\cdot)$ vanish at $\theta=\theta_o$.

The proof is the same as that of Lemma 5.1, with the change $h \to -k$ and the fact that, in the analogues of (5.23), (5.29), we integrate from θ to θ_o.

Comment 6.1: The following estimate holds: (using (6.8))

$$\varphi_{sR}(\theta) = -\frac{7}{24}\int_{\theta_a}^{\theta} k^2R_o^2\, d\theta' + O(T_{oR}^{-1/3}) \equiv \varphi_{sR}^{(o)}(\theta;R_o) + O(T_{oR}^{-1/3}) \tag{6.12}$$

Comment 6.2: The essential point in Lemma 6.1 is the fact that the integral $\int_{\theta_a}^{\theta} k^2 d\theta'$ is unbounded as $\varepsilon \to 0$; indeed, for $\theta > \theta(t_o)$, $t_o > 0$, independent of ε:

$$\varphi_{sR}(\theta) \sim -\frac{7\sqrt{3}}{24} R_o^2 \frac{1}{\gamma^{1/3}}\int_0^\infty \frac{e^{-2q}}{q^{2/3}}\, dq = -\frac{7\sqrt{3}}{48} R_o^2 \frac{2^{2/3}}{\gamma^{1/3}}\Gamma(1/3) \tag{6.13}$$

At $\theta=\theta_o$, $v_R(\theta) = O(\varepsilon^{3/16 + k\pi/2})$. Recalling Theorem 4.1, (cf. eqn. (4.3)), a convenient form of $\mathbb{P}_R$ is:

$$\mathbb{P}_R^o : u_R(0),\ du_R/d\tau(0) \longrightarrow R(\theta_o),\ \theta_o + \varphi(\theta_o) \tag{6.14}$$

with R, φ of (6.6-7). Indeed, apart from the small corrections $A_1(\varepsilon), A_2(\varepsilon)$, eqns. (3.30), (3.31), the right hand side may be directly compared with the initial values Λ, Ψ_o (cf. (5.2),(5.3)).

Now, recalling (5.51), we are in fact interested only in the restriction of $\mathbb{P}_R^o$ to a disk of radius $\varepsilon^{k\pi}$ around $(-\Delta\gamma, -\Delta'\gamma)$. Let $R_{oo}(\varepsilon)$, $\varphi_{oo}(\varepsilon)$ be the values of $R(\theta_o)$, $\varphi(\theta_o)$ corresponding to the solution of eqn. (6.4) which is such that $v_R(\theta(\tau))$, eqn. (6.3), obeys $v_R(\tau=0)$, $dv_R/d\tau(\tau=0) = \varepsilon^{1/8}(-\Delta\gamma, -\Delta'\gamma)$. We expect then that the mapping $\mathbb{P}_R^o$ is given to a first approximation by:

$$\mathbb{P}_R^o(u_R(0), du_R/d\tau(0)) \sim (R_{oo}(\varepsilon), \varphi_{oo}(\varepsilon)) + D\mathbb{P}_R^o(-\Delta\gamma, -\Delta'\gamma)(u_L(0), du_L/d\tau(0)) \tag{6.15}$$

This expectation requires more precision, since both $\varphi_{oo}(\varepsilon)$ and elements of the 2×2 matrix $D\mathbb{P}^o_R(-\Delta\gamma,-\Delta'\gamma)$ diverge as $\varepsilon\to 0$ (as a consequence of (6.13)). In the following, we discuss the approximation (6.15) in more detail.

To this end, we write $\mathbb{P}^o_R \equiv \mathbb{P}_{R2}\circ T\circ \mathbb{P}_{R1}$ to designate the three successive transformations:

$$(u_R(0), du_R/d\tau(0)) \xrightarrow{\mathbb{P}_{R1}} (u_R(\tau=T_{OR}), du_R/d\tau(T_{oR})) \xrightarrow{T} (R(\theta_a), \varphi(\theta_a))$$
$$\xrightarrow{\mathbb{P}_{R2}} (R(\theta_o), \theta_o+\varphi(\theta_o)) \tag{6.16}$$

Clearly:

$$D\mathbb{P}^o_R = D\mathbb{P}_{R2}\cdot DT\cdot D\mathbb{P}_{R1} \tag{6.17}$$

Now, $\mathbb{P}_{R1}$ is given by the evolution of $u_R(0)$, $du_R/d\tau(0)$ under:

$$\frac{d^2u_R}{d\tau^2} + 2\gamma\frac{du_R}{d\tau} + 3\eta_R^2 u_R + 3\eta_R u_R^2 + u_R^3 = 0 \tag{6.18}$$

with $\eta_R = X_R\varepsilon^{-1/8}$. The map $\mathbb{P}_{R1}$ is unknown in detail; $D\mathbb{P}_{R1}$ is obtained from the values and the derivatives at T_{OR} of two solutions of the first variation to (6.18): (with initial conditions $(\delta u, d\delta u/d\tau)(\tau=0) = (1,0),(0,1)$)

$$\frac{d^2\delta u}{d\tau^2} + 2\gamma\frac{d\delta u}{d\tau} + 3(\eta_R+u_R)^2\delta u = 0 \tag{6.19}$$

Since the solutions of (6.18) are bounded on bounded τ intervals, uniformly in ε, so are those of (6.19). Clearly, $\det D\mathbb{P}^1_R = \exp(-2\gamma T_{oR}) \sim 1$.

The transformation T has $\det DT = -3^{-1/2}R^{(o)}(T_{OR})$, where $R^{(o)}(T_{OR})$ refers to the solution of (6.18) starting at $(-\Delta\gamma,-\Delta'\gamma)$ at $\tau=0$. Since $|\Delta\gamma|+|\Delta'\gamma| \neq 0$, $R^{(o)}(T_{OR})$ is nonvanishing and bounded, even as $\varepsilon\to 0$. The matrix elements of DT are also bounded as $\varepsilon\to 0$.

Finally, we turn to $\mathbb{P}_{R2}$, eqn. (6.16). Since $\varphi_{sR}(\pi/2)\sim\gamma^{-1/3}$ as $\varepsilon\to 0$, it is convenient to consider first:

$$\widetilde{\mathbb{P}}_{R2} : (R(\theta_a), \varphi(\theta_a)) \longrightarrow (R(\theta_o), \bar{\varphi}(\theta_o)) \tag{6.20}$$

with $\bar{\varphi}(\theta_o) \equiv \bar{\varphi}_o$ of (6.10). Let $\bar{\varphi}_{oo}(\varepsilon)$ be the value corresponding to the solution of (6.18) starting at $(-\Delta\gamma,-\Delta'\gamma)$. We denote further by $R^{(o)}(\theta)$, $\varphi^{(o)}(\theta)$, the (R,φ) values corresponding to this solution and $R_a^{(o)} = R^{(o)}(\theta_a)$, $\varphi_a^{(o)} = \varphi^{(o)}(\theta_a)$. Let R_a, φ_a be the values of (R,θ) at θ_a of the neighboring solutions. Let further:

$$\bar{\varphi}(\theta) = \varphi(\theta) - \varphi_{sR}(\theta) \tag{6.21}$$

$\bar{\varphi}_{oo} = \bar{\varphi}^{(o)}(\theta_o)$. We can state the following:

Lemma 6.2: (i) The functions $R_{oo}(\varepsilon), \bar{\varphi}_{oo}(\varepsilon)$ are continuous at $\varepsilon = 0$; (ii) The derivatives $\partial R/\partial R_a(\theta), \partial\bar{\varphi}/\partial R_a(\theta)$, etc., are continuous at $\varepsilon = 0$, uniformly on any interval $[\theta_a, \theta_o(\varepsilon_o)]$, $\varepsilon_o > 0$; (iii) The elements of $D\mathbb{P}_{R2}(R_a^{(o)}, \varphi_a^{(o)})$ are continuous at $\varepsilon = 0$; (iv) The derivatives $\partial^p R/\partial\varphi_a^k\, \partial R_a^\ell(\theta)$, $\partial^p\bar{\varphi}/\partial\varphi_a^k\, \partial R_a^l(\theta)$, $k + l = p > 2$ are $O(\gamma^{(2-p)/3})$ uniformly in θ; if $p = 2$, they are $O(\ln\frac{1}{\varepsilon})$; (v) The derivatives $\partial R/\partial\beta_o, \partial\bar{\varphi}/\partial\beta_o$ are continuous at $\varepsilon = 0$, if $\beta_o > \beta_m > 0$, with β_o of (5.53).

The proof is given in Appendix E.

From Lemma 6.2(ii),(iii) it follows that $\tilde{\mathbb{P}}_R \equiv \tilde{\mathbb{P}}_{R2} \circ T \circ \mathbb{P}_{R1}$ is given by:

$$\tilde{\mathbb{P}}_R(u_R(0), du_R/d\tau(0)) \equiv (R_{oo}(\varepsilon), \bar{\varphi}_{oo}(\varepsilon)) + \varepsilon^{k\sqrt{\pi}}(D\tilde{\mathbb{P}}_R)(-\Delta\gamma, -\Delta'\gamma)(U_L(0), dU_L/d\tau(0)) + O(\varepsilon^{2k\sqrt{\pi}-\sigma}) \qquad (6.22)$$

for any $\sigma > 0$. We can now compute $\mathbb{P}_R(u_R(0), du_R/d\tau(0))$. For clarity, we write in (6.11) $\varphi_{sR}(\theta; R_a, \varphi_a)$. Then:

$$\varphi(\theta_o) = \bar{\varphi}(\theta_o) + \varphi_{sR}(\theta; R_a, \varphi_a) = \bar{\varphi}_{oo}(\varepsilon) + O(\varepsilon^{k\sqrt{\pi}}) + \varphi_{sR}(\theta; R_a^{(o)}, \varphi_a^{(o)}) + $$
$$+ \varphi_{sR}(\theta_o; R_a, \varphi_a) - \varphi_{sR}(\theta_o; R_a^{(o)}, \varphi_a^{(o)}) \qquad (6.23)$$

The term with $O(\varepsilon^{k\sqrt{\pi}})$ is obtained from (6.22); the last difference may be written:

$$\Delta\varphi_{sR}(\theta_o) = -\frac{7}{12}\int_{\theta_a}^{\theta} k^2 R^{(o)}(\theta')\, \varepsilon^{k\sqrt{\pi}}\left(\frac{\partial R(\theta')}{\partial u_R(0)} U_L(0) + \frac{\partial R(\theta')}{\partial u_R'(0)} U_L'(0) + O(\varepsilon^{k\sqrt{\pi}})\right)d\theta' \qquad (6.24)$$
$$= -\frac{7\sqrt{3}}{12} R_{oo}(\varepsilon)\frac{\varepsilon^{k\sqrt{\pi}}}{\gamma^{1/3}}\left(\int_0^\infty \frac{e^{-2q}}{q^{2/3}}dq\right)\left[J_{R1} U_L(0) + J_{R2} U_L'(0)\right] + O(\varepsilon^{k\sqrt{\pi}+s}\gamma^{-1/3})$$

for some $s > 0$ (see below). In (6.24), $U_L(0)$, $U_L'(0)$ are given in (5.40) and J_{R1}, J_{R2} are the matrix elements in the first row of the 2×2 matrix $D\tilde{\mathbb{P}}_R(-\Delta\gamma, -\Delta'\gamma)$. The justification of the last equality in (6.24) is as follows: using (6.17), it is clear that:

$$|\partial R(\theta')/\partial u_R(0) - J_{R1}| < \text{const}\Big(|\partial R/\partial R_a(\theta') - \partial R/\partial R_a(\theta_o)| + |\partial R/\partial\varphi_a(\theta') - \partial R/\partial\varphi_a(\theta_o)|\Big) \qquad (6.25)$$

where $\partial R/\partial R_a(\theta_o)$ is a matrix element of $D\tilde{\mathbb{P}}_{R2}(R_a^{(o)}, \varphi_a^{(o)})$. Using (E.2) and (E.7), we can majorize:

$$\left|\frac{\partial R}{\partial R_a}(\theta) - \frac{\partial R}{\partial R_a}(\theta_o)\right| < \text{const}\int_\theta^{\theta_o} k^4 d\theta' \int_{\theta_a}^{\theta'} k^2\, d\theta'' < C\theta^{-1/4} \qquad (6.26)$$

A tighter bound is valid for the second term in (6.25). Finally:

$$\int_{\theta_a}^{\theta} k^2 R(\theta')\, \varepsilon^{k\pi} \left| \frac{\partial R}{\partial R_{oR}}(\theta) - \frac{\partial R}{\partial R_{oR}}(\theta_o) \right| d\theta' < C_1\, \varepsilon^{k\pi} \ln \frac{1}{\varepsilon} \tag{6.27}$$

which justifies (6.24). From Appendix E, (eqn. (E.10) ff.), if we set $\varepsilon = 0$ in $R_{oo}(\varepsilon)$ in (6.22), the leading correction is still $O(\varepsilon^{k\pi})$, if k, eqn. (1.17) obeys $k < 1/7\pi$ (this is true for the bifurcation region $k = 1/8\pi$).

We can now summarize by placing together (3.39), (5.51), (6.22), (6.23), (6.24):

<u>Theorem 6.1</u>: The Poincaré map $\mathbb{P}$, eqn. (3.39) may be written:

$$\Lambda \longrightarrow R_{oo}(\varepsilon = 0) + O(\varepsilon^{k\pi}) \tag{6.28}$$

$$\begin{aligned} \Psi_o \longrightarrow\ & \pi + \theta_a + \bar{\varphi}_{oo}(\varepsilon = 0) + \varphi_{sR}(\theta; R_{oo}, \varphi_{oo}) - \\ & - \frac{7}{24}\, 3^{3/4}\, 2^{2/3}\, \Gamma(1/3)\, \Lambda\, R_{oo}(0) \frac{\varepsilon^{k\pi}}{\gamma^{1/3}} \Big[\Big(J_{R1}(\varepsilon = 0) V_1(0) + J_{R2}(\varepsilon = 0) \frac{dV_1}{d\tau}(0) \Big) \cos(-\theta_{oL} + \tilde{\varphi}_o) \\ & - \Big(J_{R1}(\varepsilon = 0)\, V_2(0) + J_{R2}(\varepsilon = 0) \frac{dV_2}{d\tau}(0) \sin(-\theta_{oL} + \tilde{\varphi}_o) \Big) \\ & + O(\varepsilon^{k\pi + s}\, \gamma^{-1/3}) \end{aligned} \tag{6.29}$$

Indices "R","L" have been placed on $\theta_{oR} \equiv \Theta_R(\pi/2)$, θ_{oL}.

<u>Comment 6.3</u>: Eqn. (6.29) makes it plain that changes in the behaviour of occur if $\beta_o \sim 1$ as $\varepsilon \to 0$ (cf. (5.53)).

There follows a statement on the derivatives of $\mathbb{P}$ with respect to Λ, Ψ_o, β_o:

<u>Theorem 6.2</u>: If $0 < \beta_o < \beta_M$, the mapping $\mathbb{P}$ has derivatives of any order with respect to Λ, Ψ_o, which are bounded as $\varepsilon \to 0$. It has a derivative with respect to β, which is also bounded as $\varepsilon \to 0$, if, in addition, $0 < \beta_m < \beta_o$.

<u>Proof</u>: Clearly:

$$D\mathbb{P}_R = D\tilde{\mathbb{P}}_R + \begin{pmatrix} 0 & 0 \\ J_{s1}(\varepsilon) & J_{s2}(\varepsilon) \end{pmatrix} \tag{6.30}$$

with $J_{s1} = (\partial \varphi_{sR} / \partial u_R(0))(\theta_o) = O(\gamma^{-1/3})$, $J_{s2} = (\partial \varphi_{sR} / \partial u'_R(0))(\theta_o) = O(\gamma^{-1/3})$.

Now:

$$D\mathbb{P} = D\mathbb{P}_R \cdot D\mathbb{P}_L \tag{6.31}$$

and the elements of $D\mathbb{P}_L$ are $O(\varepsilon^{k\pi})$ (Theorem 5.2). This shows the boundedness of the first derivatives. For the second derivatives, $x, y \in R^2$:

$$\begin{aligned} D^2\mathbb{P}(x,y) = D^2\mathbb{P}_R(D\mathbb{P}_L x, D\mathbb{P}_L y) + D\mathbb{P}_R\, D^2\mathbb{P}_L(x,y) = O(\gamma^{-1/3}\varepsilon^{2k\pi}) + O({}^{-1/3} \\ + O(\gamma^{-1/3}\varepsilon^{k\pi}) = O(1) \end{aligned} \tag{6,32}$$

using Lemma 6.2 and Theorem 5.2. The proof for the higher derivatives is analogous.

Concerning the evaluation of the derivatives of $\mathbb{P}$ with respect to Λ, ψ_o, we have the following:

Lemma 6.3: Let $\mathbb{P}^o(\Lambda, \psi_o; \beta_o)$ be the explicit part of (6.28), (6.29). The derivatives of $\mathbb{P}$ with respect to Λ, ψ_o are given to $O(\varepsilon^s)$, $s > 0$, by those of $\mathbb{P}^o(\Lambda; \psi_o; \beta_o)$ as $\varepsilon \to 0$, if $0 < \beta_o < \beta_M$. If also $\beta_o > \beta_m > 0$, $\partial\mathbb{P}/\partial\beta_o$ is given to $O(\varepsilon^s)$ by $\partial\mathbb{P}^o/\partial\beta_o$.

Proof: We write:

$$\mathbb{P}_R(u_L(0), u_L'(0)) = C_R(\varepsilon; \gamma) + \mathbb{P}_{R1}(u_L(0), u_L'(0)) + \overline{\mathbb{P}}_R(u_L(0), u_L'(0)) \qquad (6.33)$$

where C_R is a constant (vector), $\mathbb{P}_{R1}$ is linear in $u_L(0), u_L'(0)$ and $D\overline{\mathbb{P}}_R(0,0) = 0$. From Lemma 6.2, $\| D\overline{\mathbb{P}}_R(u_L(0), u_L'(0)) \| <$ const, so that $D\overline{\mathbb{P}}_R \cdot D\mathbb{P}_L = O(\varepsilon^{k\pi})$. Further, Theorem 5.2 and Comment 5.6 imply that the derivatives of $\mathbb{P}$ with respect to Λ, ψ_o are given to $O(\varepsilon^q)$, $q > 0$, by the derivatives of $\mathbb{P}_{R1} \circ (\varepsilon^{k\pi}\mathbb{P}_L^o)$. Now, for $0 < \beta_o < \beta_M$, $\mathbb{P}_{R1}(u_L, u_L')$ is given up to terms of $O(\varepsilon^r)$, $r > 0$, by (6.23-24). We call $\mathbb{P}_{R1}^o$ this "essential" part of $\mathbb{P}_{R1}$. Thus, the derivatives of $\mathbb{P}$ are approximated to $O(\varepsilon^s)$, $s > 0$, by those of $\mathbb{P}_{R1}^o \circ (\varepsilon^{k\pi}\mathbb{P}_L^o)$. But:

$$\mathbb{P}^o = C_o(\varepsilon; \gamma) + \mathbb{P}_{R1}^o \circ (\varepsilon^{k\pi}\mathbb{P}_L^o) \qquad (6.34)$$

which proves our statement (C_o is a constant) for the derivatives with respect to Λ, ψ_o.

Finally, $\partial\mathbb{P}_R/\partial\beta_o$ can be written similarly to (6.33) and, if $\beta_o > \beta_m$, the estimates of the higher derivatives with respect to $(u_L(0), u_L'(0))$ are the same as those of $\mathbb{P}_R$. The reasoning is otherwise unchanged.

Comment 6.4: We also notice:

$$\det D\mathbb{P}(\Lambda, \psi_o) = \frac{\Lambda}{P_\Lambda(\Lambda, \psi)} \varepsilon^{2k\pi} \qquad (6.35)$$

which is the Wronski identity for the variational equation. P_Λ is given by (6.28).

VII. The limit of $\mathbb{P}$ as $\varepsilon \to 0$.

We make next the meaning of (6.28-29) more transparent. First, we notice that fixed or periodic points of $\mathbb{P}$ can appear only inside an annulus $\mathcal{A}$ of radius R_{oo} and thickness $A\varepsilon^{k\pi}$, for some $A > 0$. We perform the following (ε-dependent) change of origin of ψ_o (recall (5.7), (5.35)):

$$\Psi_o = \widetilde{\Psi}_o + \theta_{oL} - \phi_{so}(\varepsilon ; R_{oo}) = T\,\widetilde{\Psi}_o \tag{7.1}$$

where we have set $\Lambda = R_{oo}$ in (5.7). Then, with an obvious notation, the $\widetilde{\Psi}$-component of $T^{-1} \circ \mathbb{P} \circ T \equiv \widetilde{\mathbb{P}}$ reads:

$$\widetilde{\Psi} \longrightarrow \pi + \theta_{oR} - \theta_{oL} + \varphi_{sR}(\theta_o ; R_{oo} ; \varphi_{oo}) + \varphi_{so}(\varepsilon ; R_{oo}) - \tag{7.2}$$

$$A\, R_{oo} \Lambda \ \frac{\varepsilon^{k\pi}}{\gamma^{1/3}} \ \cos(\widetilde{\Psi} + B + \phi_{so}(\varepsilon ; \Lambda) - \phi_{so}(\varepsilon ; R_{oo})) + O(\varepsilon^{k\pi+s}\, \gamma^{-1/3})$$

with:

$$A = \frac{7}{24}\, 3^{3/4}\, 2^{2/3}\, \Gamma(1/3)\left(\left[J_{R1} V_1 + J_{R2} \frac{dV_1}{d\tau}\right]^2 + \left[J_{R1} V_2 + J_{R2} \frac{dV_2}{d\tau}\right]^2 \right)^{1/2} \quad ,A > 0 \tag{7.3}$$

$$\tan B = \frac{J_{R1} V_2 + J_{R2} dV_2/d\tau}{J_{R1} V_1 + J_{R2} dV_1/d\tau} \tag{7.4}$$

and $B \in [-\pi/2, \pi/2]$ (or $(\pi/2, 3\pi/2)$) if $J_{R1}V_1 + J_{R2}V_2$ is positive (negative).

Now, in (7.2):

$$\theta_{oR}(\varepsilon) - \theta_{oL}(\varepsilon) = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_{-\pi/2}^{\pi/2} |\sin t|^{1/3}\, dt + \theta_{oo}(\varepsilon) \tag{7.5}$$

where $\theta_{oo}(\varepsilon) \equiv \theta_{oo}(\tau_o ; \varepsilon)$ has a finite limit as $\varepsilon \to 0$. (The latter is verified using the approximants (3.8),(3.28) to X_L, X_R).

We rewrite (7.2) as:

$$\widetilde{\Psi} \rightarrow C(\varepsilon ; \gamma ; \Lambda ; \Psi) - \beta(\varepsilon ; \gamma)\, \frac{\Lambda}{R_{oo}} \cos(\widetilde{\Psi} + B + \phi_{so}(\varepsilon ; \Lambda) - \phi_{so}(\varepsilon ; R_{oo})) \tag{7.6}$$

with:

$$C = C_o(\varepsilon ; \gamma) + O(\varepsilon^{k\pi+s}\, \gamma^{-1/3}) \tag{7.7}$$

$$\beta = \beta_o\, A\, R_{oo}^2 \tag{7.8}$$

where C_o has a clear meaning and β_o appears in (5.53). We change further variables to:

$$\widetilde{\Psi} = \chi + C_o(\varepsilon ; \gamma) \equiv S\chi \tag{7.9}$$

so that $(S^{-1} \circ \widetilde{\mathbb{P}} \circ S) \equiv \widetilde{\pi}$ re ads:

$$\chi \rightarrow \beta\, \frac{\Lambda}{R_{oo}} \cos(\chi + \widetilde{\Sigma}(\varepsilon ; \beta ; \Lambda)) + O(\varepsilon^{k\pi+s}\, \gamma^{-1/3}) \tag{7.10}$$

with:

$$\widetilde{\Sigma} = C_o(\varepsilon ; \gamma) + B + \pi + \phi_{so}(\varepsilon ; \Lambda) - \phi_{sb}(\varepsilon ; R_{oo}). \tag{7.11}$$

Let now:

$$\Sigma(\varepsilon ; \gamma) \equiv C_o(\varepsilon ; \gamma) + B + \pi \tag{7.12}$$

and $\overline{J}|$ be the mapping of the unit circle into itself given by:

$$\pi : \quad \chi \longrightarrow \beta \cos(\chi + \Sigma) \qquad (\mathrm{mod}\ 2\pi) \tag{7.13}$$

As $\varepsilon \to 0$, the mapping $\widetilde{\pi}(\varepsilon ; \beta)$, given by (7.10) (and (6.28)) has no limit because of the indefinite increase of $\tilde{\Sigma}(\varepsilon;\Lambda)$, like $\varepsilon^{-1/2}$, in (7.10). However, if we let ε_n tend to zero so that $\tilde{\beta}$ = const and $\Sigma(\varepsilon_n;\gamma) = \alpha$ (mod 2π), for a given value α, the mapping $\tilde{\pi}(\varepsilon_n,\beta)$ obviously approaches $\bar{\pi}$ of (7.13), with $\Sigma = \alpha$. Thus, the limit of $\mathbb{P}(\varepsilon;\beta_o)$ as $\varepsilon \to 0$ with β_o = const is the family $\bar{\pi}(\beta;\Sigma)$ of onedimensional mappings, with $0 \leqslant \Sigma < 2\pi$ (and $\Lambda = R_{oo}$)

Comment 7.1 : We can also let $\beta_o \to 0$ as $\varepsilon \to 0$; then $\chi = 0, \Lambda = R_{oo}$ is the (trivial) limit of $\mathbb{P}$.

In the following, we study the bifurcation structure of $\bar{\pi}$, eqn. (7.13); in the next Section., we discuss the extent to which the bifurcations of $\mathbb{P}(\varepsilon;\beta)$ may be inferred from those of $\bar{\pi}$ for small ε. We note that the map (7.13) has been recently studied (also numerically) in Ref.[24]. Eqn. (7.13) is different from the map $x_{n+1} = x_n + \tau + B \sin 2\pi x_n$ used to model a nonlinear oscillator with a limit cycle (Refs.[25,26]): the term linear in x_n is missing. We shall be concerned here only with the first few bifurcations of $\bar{\pi}$, which we describe by elementary methods.

(i) Lemma 7.1:If $\beta < 1$, $\bar{\pi}$ has a unique fixed point and is a contraction for all Σ.

This is obvious.

(ii) At every fixed Σ, if we increase β, we reach a value $\beta_s(\Sigma)$ beyond which $\bar{\pi}$ admits of three (or more) fixed points, i.e. undergoes a saddle - node bifurcation. At $\beta = \beta_s(\Sigma)$, the derivative of $\bar{\pi}$ at the fixed point χ_s is unity, i.e. the equations:

$$\chi_s = \beta_s \cos(\chi_s + \Sigma) \qquad (\text{mod } 2\pi) \tag{7.14}$$

$$1 = -\beta_s \sin(\chi_s + \Sigma) \tag{7.15}$$

hold. It follows that $\beta_s^2 = 1 + \chi_s^2 \geqslant 1$; we have $\beta_s = 1$ only if $\chi_s = 0$, i.e. only if $\Sigma = \mp\pi/2$ (mod 2π). Otherwise $\beta_s^* > 1$.

(iii) We discuss the shape of the saddle - node bifurcation curves near $\Sigma = -\pi/2$. Let $\sigma = \Sigma + \pi/2$. From (7.14-15), we deduce:

$$\tan(\chi_s + \sigma) = \chi_s \tag{7.16}$$

i.e. for small χ_s, σ :

$$\chi_s(\sigma) = -(3\sigma)^{1/3}. \tag{7.17}$$

It follows that:

$$\beta_s(\sigma) \simeq 1 + \frac{(3\sigma)^{2/3}}{2} \tag{7.18}$$

i.e. $\beta_s(\Sigma)$ has a cusp at $\Sigma = -\pi/2 \pmod{2\pi}$.

(iv) If, at the fixed point χ_F, it is true that $(d\mathcal{T}/d\chi)(\chi_F) = -1$, the map $\mathcal{T}$ has a flip bifurcation, if certain transversality conditions are obeyed (see, e.g. Ref.[8], §3.5); the latter are simply $\chi_F^2 + 2 \neq 0$, $\chi_F^2 + 2/3 \neq 0$ in our case. The analogues of (7.14-15) are:

$$\chi_F = \beta_F \cos(\chi_F + \Sigma) \tag{7.19}$$

$$1 = \beta_F \sin(\chi_F + \Sigma) \tag{7.20}$$

Again, it follows that $\beta_F(\Sigma) > 1$, unless $\chi_F = 0$, i.e. $\Sigma = \pi/2 \pmod{2\pi}$.

(v) Let now $\sigma = \Sigma - \pi/2$. From (7.19-20) it follows that:

$$\tan(\chi_F + \sigma) = -\chi_F \tag{7.21}$$

i.e.

$$\chi_F(\sigma) = -\sigma/2 + 0(\sigma^3) \tag{7.22}$$

and

$$\beta_F(\sigma) \cong 1 + \frac{\sigma^2}{8} \tag{7.23}$$

Thus, $\beta_F(\Sigma)$ has a quadratic minimum at $\Sigma = \pi/2$.

(vi) Clearly, the flip bifurcation curves $\beta_F(\Sigma)$ are broader than the saddle node curves $\beta_s(\Sigma)$. We can be more precise about this. At the point $(\Sigma_o, \hat{\beta}_o)$ of intersection of $\beta_F(\Sigma)$ with $\beta_s(\Sigma)$, eqns. (7.14-15),(7.19-20) hold for the four unknowns $\hat{\beta}_o, \Sigma_o, \chi_F^o, \chi_s^o$. We can assume $\hat{\beta}_o < \pi$ (as will be apparent). From (7.14-15),(7.19-20), it follows that: $1 + (\chi_F^o)^2 = 1 + (\chi_s^o)^2$ at $(\Sigma_o, \hat{\beta}_o)$, i.e.

$$\chi_F^o = \pm \chi_s^o \pmod{2\pi} \tag{7.24}$$

and, since $\hat{\beta}_o < \pi$, $|\chi_F^o|, |\chi_s^o| < \pi$. Clearly, also:

$$\cot(\chi_F^o + \Sigma_o) = \chi_F^o \tag{7.25}$$

$$\cot(\chi_s^o + \Sigma_o) = -\chi_s^o \tag{7.26}$$

The possibility $\chi_s^o = \chi_F^o$ is ruled out by (7.15), (7.20). Thus $\chi_s^o = -\chi_F^o$ and therefore:

$$\chi_F^o + \Sigma_o = \chi_s^o + \Sigma_o \pmod{\pi} \tag{7.27}$$

Then, at the intersection points of $\beta_F(\Sigma)$ with $\beta_s(\Sigma)$:

$$\chi_s^o = \pm \pi/2 \quad , \chi_F^o = \mp \pi/2 \quad , \quad \hat{\beta}_o = (1 + \pi^2/4)^{1/2} \tag{7.28}$$

The possibility $\chi_s^o = -\pi/2$ leads to $\tan \Sigma_o = -\pi/2$ and eqns. (7.15),(7.20) require $\Sigma_o = -\operatorname{atan} \pi/2$, with $-\pi/2 < \Sigma_o < \pi/2$. If $\chi_s^o = \pi/2$, (7.15),(7.20) require: $\Sigma_o = \pi + \operatorname{atan} \pi/2$, $\pi < \Sigma_o < 3\pi/2$. It follows that the ratio ρ of the widths (measured at the intersection points, in units of $\Gamma^{1/3}$) of $\beta_s(\Sigma)$ and $\beta_F(\Sigma)$ is:

$$\rho = \frac{\pi - 2 \operatorname{atan} \pi/2}{\pi + 2 \operatorname{atan} \pi/2} \cong 0.22 \tag{7.29}$$

(vii) We find next at every Σ an interval of values of β, $0 < \beta < \beta_e(\Sigma)$, in which the invariant sets of $\overline{\mathcal{J}}$ can be completely described. The argument avoids the theory of onedimensional mappings of Ref.[9], but uses the notion of Schwarz derivative:

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{7.30}$$

in the manner of the proof of Theorem 6.3.1, Ref.[8] or Theorem II.4.1, Ref.[9].

<u>Lemma 7.2</u>: Let $\beta_{2u}(\Sigma)$ be the second positive root of the equation:

$$\overline{\mathcal{J}}(\beta;\chi=\beta) = \beta\cos(\beta+\Sigma) = -\Sigma \tag{7.31}$$

if $\Sigma < 0$ and the first positive root of (7.31) if $\Sigma > 0$ $(-\pi < \Sigma < \pi)$. Let $\beta_{2d}(\Sigma)$ be the second positive root of:

$$\overline{\mathcal{J}}(\beta;\chi=-\beta) = \beta\cos(\beta-\Sigma) = -\pi-\Sigma \tag{7.32}$$

if $\Sigma < 0$ and the first positive root of

$$\overline{\mathcal{J}}(\beta;\chi=-\beta) = \beta\cos(\beta-\Sigma) = \pi-\Sigma \tag{7.33}$$

if $\Sigma > 0$. Let:

$$\beta_e(\Sigma) = \min[\pi, \beta_{2u}(\Sigma), \beta_{2d}(\Sigma)] . \tag{7.34}$$

Then, if $0 < \beta < \beta_e(\Sigma)$, the invariant sets of $\overline{\mathcal{J}}$ consist of at most three fixed points and two pairs, each of period two.

The proof is displayed in Appendix F.

<u>Comment 7.2</u>: Eqns. (7.31), (7.32) also admit of a first positive root, $\beta_{1u} = -\Sigma$, if $\Sigma < 0$ in (7.31) and $\beta_{1d} = \pi + \Sigma$ in (7.32). At these values of β, $\overline{\mathcal{J}}$ admits of a superstable[9] fixed point, at $\chi = -\Sigma$ (i.e. lying on the positive maximum), $\chi = -\pi - \Sigma$, in turn.

Comment 7.3: The other roots in $\beta > 0$ of (7.31-33) correspond to superstable period two orbits, through the maximum or minimum of $\mathcal{J}(\chi)$.

(viii) The set $\mathcal{J}(\chi;\beta;\Sigma = -\pi/2)|_{[0,\pi]}$ makes up, for $\pi/2 < \beta < \pi$ a full family of unimodal maps, in the sense of Ref.[9]. For such a family, there exists an increasing sequence of values of $\beta : \beta_1 < \beta_2 < \ldots$ at which $\mathcal{J}|_{[0,\pi]}$ admits of superstable orbits of period 2^p. This sequence accumulates to $\beta_c < \pi$. For $\pi > \beta > \beta_c$, the motion has no stable periodic orbits (is chaotic) for a large set of values of β (see Ref.[9]).

Clearly, the same situation is expected to hold at neighboring values of Σ. This closes the discussion of the limiting mapping $\mathcal{J}$, eqn. (7.13).

VIII. The bifurcations of periodic solutions at large Γ ($\varepsilon \to 0$).

Clearly, the bifurcation structure of $\mathcal{J}$ is periodic as a function of Σ, at fixed β. We may take Δ, Γ as parameters of $\tilde{\mathcal{J}}$. From (7.12), (7.7), (7.5), we obtain:

$$\Sigma \sim \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_{-\pi/2}^{\pi/2} |\sin t|^{1/3}\, dt + \frac{A}{\gamma^{1/3}} \simeq \frac{B}{\sqrt{\varepsilon}} = B\Gamma^{1/3} \tag{8.1}$$

since $\gamma^{1/3} \sim \varepsilon^{1/8}$ and $\beta = \hat{\beta}$ = const means $\Delta = \Delta_c(\Gamma)$ of (1.12) with:

$$C_o = -\frac{1}{3\pi} \ln \frac{1}{12\pi} + \frac{1}{\pi} \ln (A R_{oo}^2) - \frac{1}{\pi} \ln \hat{\beta} \tag{8.2}$$

The preceding Section tells that, if Γ is large, $\mathcal{J}(\Gamma;\Delta)$ possesses an alternating sequence of saddle - node and odd periodic - simply periodic bifurcation curves, so that their maxima lie on the line $\beta = 1$, with a spacing equal to π in the variable Σ, i.e. given by (1.3) in $\Gamma^{1/3}$. If $\beta < \min(\beta_F(\Sigma), \beta_s(\Sigma))$ (in particular, if $\beta < 1$ or if $\beta \to 0$ as $\Gamma \to \infty$), $\mathcal{J}(\Gamma;\Delta)$ has a unique fixed point. We expect that, for Γ large, the mapping $\tilde{\mathcal{J}}$, eqns. (7.10), (6.28) - which is equivalent to $\mathbb{P}$, eqn. (3.39) - admits of the same bifurcation structure. We write in the following $\mathbb{P}$ for all its equivalent forms, including $\tilde{\mathcal{J}}$.

To this end, recalling (7.10), (7.12), (6.28), we rewrite $\mathbb{P}$ in the form:

$$\Lambda \to R_{oo} + O(\varepsilon^{k\pi}) \tag{8.3}$$

$$\chi \to \beta \frac{\Lambda}{R_{oo}} \cos(\chi + \Sigma + \Phi_{sL}(\varepsilon;\Lambda) - \Phi_{sL}(\varepsilon;R_{oo})) + O(\varepsilon^{k\pi + s}\gamma^{-1/3}) \tag{8.4}$$

We keep β, Σ as fixed parameters in (8.3),(8.4) and study the transfer of the invariant sets from $\mathbb{J}(\beta;\Sigma)$ to $\mathbb{P}(\beta;\Sigma;\varepsilon)$,eqns. (8.3-4) for small ε. Keeping Σ fixed means we let a sequence ε_n tend to zero, with $\Sigma(\varepsilon_n, \Upsilon(\beta, \varepsilon_n))$ fixed (mod 2π).

Comment 8.1: Using Lemma 6.3 and computing explicitly the derivatives of $\mathbb{P}^{\circ}(\beta;\Sigma;\varepsilon_n)$ we conclude that $(\partial P_\chi/\partial\chi)'_\Sigma$ approaches $(\partial\mathbb{J}/\partial\chi)_\Sigma$ as $\varepsilon_n \to 0$, whereas $(\partial P_\chi/\partial\Lambda)_\Sigma \sim \mathbb{J}(\chi;\Sigma)/R_{oo}$ - const$\cdot(d\mathbb{J}/d\chi)(\chi;\Sigma)/(\ln 1/\varepsilon)^{2/3}$ for small ε, $0<\beta<\beta_M$. Also, $\partial P_\Lambda/\partial\chi \sim \partial P_\Lambda/\partial\Lambda \sim 0(\varepsilon^{k\mathbb{J}})$, by Lemma 6.3.

In the above and in the following we write $\mathbb{P} = (P_\Lambda, P_\chi)$ for the two components of $\mathbb{P}$. (cf.(6.35)).

There now follow a number of simple statements, relating the behaviour of $\mathbb{J}$ with that of $\mathbb{P}$.

Lemma 8.1: Let χ_o be a fixed point (or periodic with period p) of $\mathbb{J}(\beta;\Sigma;\chi)$ for which $\partial\mathbb{J}/\partial\chi \neq 1$ (or $\partial\mathbb{J}^p/\partial\chi \neq 1$). If $0<\beta<\beta_M$, then, for ε small enough, $\mathbb{P}(\beta;\Sigma;\varepsilon)$ also has a fixed point $\bar{R}_o, \bar{\chi}_o$ (or a periodic point of period p, in turn) in $0<\Lambda<\Lambda_M$ (cf. eqn. (4.3)):

$$|\bar{R}_o - R_{oo}(\varepsilon=0)| = 0(\varepsilon^{k\mathbb{J}}) \quad , \quad |\bar{\chi}_o - \chi_o| = 0(\varepsilon^s + \varepsilon^{k\mathbb{J}}) \tag{8.5}$$

Proof: We use Lemma 6.3. At each fixed χ, $\partial P_\Lambda/\partial\Lambda = 0(\varepsilon^s)$ so that $P_\Lambda - \Lambda$ is, for each χ and small ε, monotonic in Λ, in $0<\Lambda<\Lambda_M$. The unique root $\Lambda = \Lambda(\chi;\varepsilon)$ of $P_\Lambda - \Lambda = 0$ in this interval is such that $|\Lambda(\chi;\varepsilon) - R_{oo}| = 0(\varepsilon^{k\mathbb{J}})$ and $(\partial\Lambda/\partial\chi)_\varepsilon = 0(\varepsilon^s)$, since $\partial P_\Lambda/\partial\chi = 0(\varepsilon^s)$. Substituting this into (8.4), we obtain using also:

$$\varphi_{so}(\varepsilon;\Lambda(\chi;\varepsilon)) - \varphi_{so}(\varepsilon;R_{oo}) = 0\left(\frac{\varepsilon^{2k\mathbb{J}}}{\gamma^{1/3}} \frac{1}{(\ln\frac{1}{\varepsilon})^{2/3}}\right) \tag{8.6}$$

that:

$$|\partial P_\chi/\partial\chi - \partial\mathbb{J}/\partial\chi| = 0(\varepsilon^q) \quad , \quad q>0 \tag{8.7}$$

Thus, for χ near χ_o and ε small enough, $P_\chi - \chi$ is monotonical in χ and changes sign. From (8.4), its unique root there, $\bar{\chi}_o$, obeys (8.5). This ends the proof.

Lemma 8.2: Assume that the fixed and periodic points of period $p \leq p_o$ of $\mathbb{J}(\beta;\Sigma)$ are such that $(\mathbb{J}^p)'_\chi \neq 1$. Then, if ε is small enough, they are in one - to - one correspondence with the fixed and periodic points (with period less than p_o) of $\mathbb{P}(\beta;\Sigma;\varepsilon)$.

Proof: From Lemma 8.1, if $\varepsilon < \varepsilon_o$, there exists a unique root of $I - \mathbb{P}$ (or $I - \mathbb{P}^p$) lying in the closed set $[\chi_i - \delta, \chi_i + \delta] \times [R_{oo} - C\varepsilon_o^{k\pi}, R_{oo} + C\varepsilon_o^{k\pi}]$, where χ_i are the fixed points of $\mathcal{J}$ (or $\mathcal{J}^p$). Now, $I - \mathbb{P}$ tends to $I - \mathcal{J}$ (with an obvious notation) uniformly as $\varepsilon \to 0$ on $[0,2\pi]\times[0<\Lambda<\Lambda_M]$. Thus, there exists a compact set $A \subset [0,2\pi]\times[|\Lambda|<\Lambda_M]$, containing the complement of the above sets, where $I - \mathbb{P}$ cannot vanish for small ε. This ends the proof.

From Lemmas 8.1, 8.2, we deduce that, if $\mathcal{J}$ has a unique fixed point χ_o, the mapping $\mathbb{P}(\beta;\Sigma;\varepsilon)$ also has, for small ε, a unique fixed point $(\Lambda_{o\varepsilon}, \chi_{o\varepsilon})$. To exclude that, for large p, $\mathbb{P}^p(\beta;\Sigma;\varepsilon)$ might have other invariant sets ($p \to \infty$ as $\varepsilon \to 0$), we state:

Lemma 8.3: Let $\delta > 0$; if $\mathcal{J}$ has a fixed point χ_o, such that, for all $|\chi|<\beta$, $|\mathcal{J}(\chi) - \chi_o| < (1-\delta)|\chi - \chi_o|$, there exists $\varepsilon_o(\delta)$, independent of Σ, so that, if $\varepsilon_n < \varepsilon_o(\delta)$, $\mathbb{P}(\beta;\Sigma;\varepsilon_n)$ has no other invariant sets in D: $|\Lambda|<\Lambda_M$ apart from the fixed, attracting point $(\Lambda_{o\varepsilon}, \chi_{o\varepsilon})$ corresponding to χ_o by Lemma 8.2.

Proof: From Comment 8.1, it is clear that, if $\varepsilon_n < \varepsilon_1(\delta)$, appropriately small:

$$|P_{\Lambda\varepsilon}(\Lambda;\chi) - \Lambda_{o\varepsilon}| < C\varepsilon^{k\pi}(|\Lambda - \Lambda_{o\varepsilon}| + |\chi - \chi_{o\varepsilon}|) \tag{8.8}$$

$$|P_{\chi\varepsilon}(\Lambda;\chi) - \chi_{o\varepsilon}| < A|\Lambda - \Lambda_{o\varepsilon}| + (1 - 3\delta/4)|\chi - \chi_{o\varepsilon}| \tag{8.9}$$

for all $\Lambda, \chi \in D$. We show first that, under successive iterations of $\mathbb{P}$, D is mapped into a rectangle R_1 of size $|\Lambda - \Lambda_{o\varepsilon}| < 4\pi C\varepsilon^{k\pi} \equiv M_1\varepsilon^{k\pi}$, $|\chi - \chi_{o\varepsilon}| < (16\pi AC/\delta)\cdot\varepsilon^{k\pi} \equiv K_1\varepsilon^{k\pi}$. Indeed, under $\mathbb{P}^2$, D is mapped into a ring $|\Lambda - \Lambda_{o\varepsilon}| < 4\pi C\varepsilon^{k\pi}$, if $\varepsilon^{k\pi} < \pi/[C(\Lambda_M + 2\pi)]$, which is further mapped into itself under $\mathbb{P}$, if $\varepsilon^{k\pi} < 1/(4C)$. Now, if $|\chi - \chi_{o\varepsilon}| > K_1\varepsilon^{k\pi}/2$, $|P_{\chi\varepsilon}(\Lambda;\chi) - \chi_{o\varepsilon}| < (1-\delta/4)|\chi - \chi_{o\varepsilon}|$, i.e. $|\chi - \chi_{o\varepsilon}|$ is contracted. If $|\chi - \chi_{o\varepsilon}| < K_1\varepsilon^{k\pi}/2$, then $|P_{\chi\varepsilon}(\Lambda;\chi) - \chi_{o\varepsilon}| < K_1\varepsilon^{k\pi}$, if $\delta < 1/2$. It is now easy to verify for $p > 1$ that the rectangle R_p of size $|\Lambda - \Lambda_{o\varepsilon}| < M_p\varepsilon^{pk\pi}$, $|\chi - \chi_{o\varepsilon}| < K_p\varepsilon^{pk\pi}$ is mapped into R_{p+1} if $q = 2C(1+A+4A/\delta)$, $M_p = q^{p-2}T_1C$, $K_p = q^{p-2}T_1\,4AC/\delta$, $T_1 = 2K_1 + 2M_1(1+A)$, $\varepsilon^{k\pi} < 1$. The rectangles converge to zero if $q\varepsilon^{k\pi}$ is sufficiently small. This ends the proof.

Comment 8.2: We may obviously require in Lemma 8.3 $|\mathcal{J}^2(\chi) - \chi_o| < (1-\delta)|\chi - \chi_o|$ instead of referring to $\mathcal{J}$. The conclusionsis the same since $\mathbb{P}^2$ may also be estimated as in (8.8-9). This is of relevance if $\beta_1(\Sigma) < \beta < \beta_F(\Sigma)$, cf. Lemma 7.2.

Lemma 8.4: Let $\beta<\beta_e(\Sigma)$, eqn. (7.34). Assume that, at the fixed points of $\mathcal{J}$ or $\mathcal{J}^2$, $|\mathcal{J}'_\chi-1|>\delta$, $|(\mathcal{J}^2)'_\chi-1|>\delta$, for some $\delta>0$. There exists then $\varepsilon_o(\delta)$ so that, if $\varepsilon_n<\varepsilon_o(\delta)$, $\mathbb{P}(\beta;\Sigma;\varepsilon_n)$ has no other invariant sets apart from those corresponding to $\mathcal{J}$ by Lemma 8.2.

Proof: By Lemma 8.2, if $\varepsilon<\varepsilon_1(\delta)$, $\mathbb{P}$ has the same number of periodic points of period $p\leq 2$ as $\mathcal{J}$, and they approach those of the latter as $\varepsilon\to 0$. For simplicity, we assume that $\mathcal{J}$ has three fixed points χ_-,χ_o,χ_+. Using the argument of Lemma 7.2, there exists $\delta_1(\delta)$, so that, if $\chi\in[\chi_o+\delta_1,\beta]\equiv I_+$, the quantity $k_+=\min|\mathcal{J}^2(\chi)-\chi_+|/|\chi-\chi_+|$ obeys $k_+<1-\delta_1$, and similarly for I_-, k_-, with obvious notations. As in Lemma 8.3, we verify that all points in $I_+\times\{|\Lambda-\Lambda_{+\varepsilon}|<C_o\varepsilon^{k\pi}\}$ are attracted under $\mathbb{P}(\varepsilon)$ to $(\Lambda_{+\varepsilon},\chi_{+\varepsilon})$ and similarly for $I_-,\Lambda_{-\varepsilon},\chi_{-\varepsilon}$. $(C_o>C(\Lambda_M+2\pi))$. Thus, we only have to show that the set $\{|\Lambda-\Lambda_{o\varepsilon}|<C_o\ \varepsilon^{k\pi}\}\times\{|\chi-\chi_{o\varepsilon}|<\delta_1\}\equiv U_\varepsilon$ does not contain other invariant sets of $\mathbb{P}$, except for $(\Lambda_{o\varepsilon},\chi_{o\varepsilon})$. If ε is small, we can assume $|\partial P_\chi/\partial\chi|>1+\delta_2>1$, $|\partial P_\chi/\partial\Lambda|<A$ in U .

Consider then the angle $W:|\Lambda-\Lambda_{o\varepsilon}|<(\delta_2/2A)|\chi-\chi_{o\varepsilon}|$, in the (Λ,χ) plane. One verifies that, if $(\Lambda,\chi)\in W$, then $|P_\chi(\Lambda;\chi)-\chi_{o\varepsilon}|>(1+\delta_2/2)|\chi-\chi_{o\varepsilon}|$. Moreover, points of $W\cap U_\varepsilon$ are mapped in W, if ε is small. Thus, if a point reaches W under iterates of $\mathbb{P}$, $|\chi-\chi_{o\varepsilon}|$ increases until it reaches the boundary of U_ε, so that there are no invariant sets in $W\cap U_\varepsilon$.

Now, we show that, under iteration of $\mathbb{P}$, every point in U_ε either reaches W after a finite number of steps or the sequence thus generated converges to $(\Lambda_{o\varepsilon},\chi_{o\varepsilon})$. Indeed, all points of U_ε that are not in W lie inside the rectangle $U_{1\varepsilon}=\{|\Lambda-\Lambda_{o\varepsilon}|<C_o\varepsilon^{k\pi}\}\times\{|\chi-\chi_{o\varepsilon}|<(3AC_o/\delta_2)\varepsilon^{k\pi}\}\subset U_\varepsilon$. As in Lemma 8.3, under p iterations of $\mathbb{P}$ those points of U_ε that do not reach W are contained in a rectangle with sides $M_p\varepsilon^{pk\pi}$, $3AM_p\varepsilon^{pk\pi}/\delta_2$, with $M_p=q^{p-1}C_o$, $q=C(1+3A/\delta_2)$. Clearly, $M_p\varepsilon^{pk\pi}\to 0$ as $p\to\infty$, if ε is small. This ends the proof.

We turn now to those values β,Σ where $\mathcal{J}(\beta;\Sigma;\chi)$ has fixed points for which $|\mathcal{J}'_\chi|$ is closedto unity.

Lemma 8.5: Let $2\pi-\delta>|\Sigma+\pi/2|>\delta>0$, $\beta>\beta_m>0$. For every $e>0$, there exists $\varepsilon_o(e,\delta)$ so that, if $\varepsilon_n<\varepsilon_o$, the set of three equations: $\mathbb{P}(\beta_s,\Sigma,\Lambda,\chi;\varepsilon_n)=$

(Λ,χ), det $(I - D\mathbb{P})(\beta_s,\Sigma,\Lambda,\chi;\varepsilon_n) = 0$ admits of a unique solution $\beta_s(\Sigma;\varepsilon)$, $\chi_s(\Sigma;\varepsilon), \Lambda_s(\Sigma;\varepsilon)$, differentiable as a function of Σ and departing from $\beta_s(\Sigma), \chi_s(\Sigma)$, R_{oo} by less than e.

Proof: First, if $|\Sigma+\pi/2| > \delta > 0, |\partial^2\mathcal{J}/\partial\chi^2(\beta_s,\Sigma,\chi_s)|, |\partial\mathcal{J}/\partial\beta(\beta_s,\Sigma,\chi_s)| > \delta_1 > 0$ (from (7.14-15)). With this, the statement follows by simply solving the set of equations mentioned above in terms of Λ, β, χ in turn, in the manner of Lemma 8.1. Intervals of monotonicity in Λ, β, χ in turn exist, for $|\chi - \chi_s(\Sigma)|$ sufficiently small and λ, β close to $R_{oo}, \beta_s(\Sigma)$ since, by Lemma 6.3, we verify that $\partial P_\chi/\partial\chi$, $\partial^2 P_\chi/\partial\chi^2$, $\partial P_\chi/\partial\beta$, $\partial^2 P_\chi/\partial\beta\partial\chi$ approach as $\varepsilon \to 0$ the corresponding derivatives of $\mathcal{J}$; thus, for ε small, the relevant quantities are certainly nonvanishing.

As in Lemmas 8.4, 8.5, we show that $\mathbb{P}$ has no other invariant sets in a neighbourhood of $\Lambda_s(\Sigma;\varepsilon), \chi_s(\Sigma;\varepsilon)$.

Lemma 8.6: At every $\Sigma, 2\pi-\delta > |\Sigma+\pi/2| > \delta > 0$ and ε small, there exists a neighbourhood $V = U_o \times U_1$ of $(\beta_s(\Sigma;\varepsilon), \Lambda_s(\Sigma;\varepsilon), \chi_s(\Sigma;\varepsilon))$: $U_o = \{|\beta-\beta_s| < A\}$, $U_1: \{|\Lambda-\Lambda_s| < C_o\,\varepsilon^{k\pi}\} \times \{|\chi-\chi_s| < B\}$, with A, B, C_o independent of ε and having the property that: if $\beta < \beta_s$, U_1 contains no invariant sets of $\mathbb{P}(\beta;\Sigma;\varepsilon)$; if $\beta > \beta_s$ the limit set consists of precisely two points; if $\beta = \beta_s$, of (Λ_s, χ_s) alone.

Proof: This is done most easily using the center manifold theorem (Ref.[27], p.28) for the map $\mathbb{P}(\beta;\Sigma;\Lambda;\chi;\varepsilon_n)$ at fixed Σ, at $(\beta_s, \Lambda_s, \chi_s)$. Let Λ', χ' be new coordinates, related linearly to $\Lambda-\Lambda_s, \chi-\chi_s$ so that the linear part of $\mathbb{P}(\beta_s)$ is diagonal. There exists then a neighbourhood V of $(\beta_s, \Lambda_s, \chi_s)$ and a function $\Lambda' = u(\beta', \chi')$ whose graph is contained in V, such that: (i) the set $M = \{\beta', \Lambda' = u(\beta',\chi'), \chi'\}$ is invariant under $\mathbb{P}$ and (ii) all points in V approach M under $\mathbb{P}$ (see Ref.[27]). $(\beta' = \beta - \beta_s)$

The magnitude of V is independent of ε, since the eigenvalues of $D\mathbb{P}$ that are different from unity stay away from the unit circle (they are $O(\varepsilon^{2k\pi})$)(see Ref.[27]). Now, according to the center manifold theorem, all invariant sets of $\mathbb{P}|_V$ are contained in the slices $\beta =$ const of the set M. The action of $\mathbb{P}$ on M is given by:

$$\chi_1' = \chi' + a(\beta-\beta_s) - b\,\chi'^2 + X(\beta-\beta_s, \chi') \tag{8.10}$$

where a,b $\neq 0$ for small ε (because $\partial^2 \mathcal{J}/\partial\chi^2 \neq 0$, $\partial\mathcal{J}/\partial\beta \neq 0$ and $u(\beta', \chi') = O(\varepsilon^{k\pi})$ as one may verify), and $X(0,0) = DX(0,0) = 0$, $X = o(|\beta - \beta_s| + \chi'^2)$. The onedimensional map (8.10) is the standard form of a saddle - node bifurcation and, if the neighbourhood V is suitably restricted, the statements of the Theorem may be directly verified (see also Refs.[28,29]).

For the flip bifurcations we have in strict analogy:

Lemma 8.7: For any $e > 0$, there exists $\varepsilon_o(e)$, so that, at every Σ, for $\varepsilon < \varepsilon_o(e)$, the set of equations $\mathbb{P}(\beta_F, \Sigma, \Lambda_F, \chi_F, \varepsilon_n) = (\Lambda_F, \chi_F)$, $\det(I + D\mathbb{P})(\beta_F, \Sigma, \Lambda_F, \Sigma_F, \varepsilon) = 0$ admits of a unique solution $\beta_F(\Sigma; \varepsilon_n), \Lambda_F(\Sigma; \varepsilon_n), \chi_F(\Sigma; \varepsilon_n)$, differentiable with respect to Σ and departing from $\beta_F(\Sigma)$, R_{oo}, $\chi_F(\Sigma)$ by less than e '. (cf. eqns.(7.19-20)). At every fixed $\varepsilon_n < \varepsilon_o(e)$, there exists a neighbourhood V of $\beta_F(\Sigma;\varepsilon_n), \Lambda_F(\Sigma;\varepsilon_n), \chi_F(\Sigma;\varepsilon_n)$ so that: (i) if $\beta \leq \beta_F(\Sigma;\varepsilon_n)$, the limit set of $\mathbb{P}$ in V consists of one fixed point; (ii) if $\beta > \beta_F(\Sigma;\varepsilon_n)$, the limit set consists of one fixed point and one stable orbit of period two.

The proof is the same as for Lemmas 8.6-7.For a related detailed discussion, see Refs.[28,29].

The analysis of the deformation of the cusp in $\beta_s(\Sigma)$ at $\Sigma = -\pi/2$ appears to be more difficult. We can only state the (rather obvious):

Lemma 8.8: If $\varepsilon, \Sigma + \pi/2, \beta - 1$, A are sufficiently small, $\mathbb{P}(\beta, \Sigma, \Lambda, \chi, \varepsilon)$ has either three or one fixed point in $\{|\Lambda - R_{oo}| < C_o\, \varepsilon^{k\pi}\} \times \{|\chi| < A\}$.

Proof: At each $\beta, \Sigma, \varepsilon$ we can solve $P_\Lambda = \Lambda$, for every χ. The solution is $\Lambda = R_{oo} + \varepsilon^{k\pi}\, u(\beta - 1, \Sigma + \pi/2, \varepsilon; \chi)$. The function u has derivatives with respect to χ and β, bounded as $\varepsilon \to 0$ (Lemma 6.3). Substituting in $P_\chi - \chi = 0$, we obtain an equation:

$$G(\chi; \beta; \varepsilon; \Sigma) = P_\chi(\beta, \Sigma, \Lambda(\beta, \Sigma, \varepsilon, \chi), \chi, \varepsilon) - \chi = 0 \tag{8.11}$$

Now,G tends with all its relevant derivatives as $\varepsilon \to 0$ to the corresponding values of $\mathcal{J}(\beta; \Sigma; \chi) - \chi$, since the derivatives of Λ are $O(\varepsilon^{k\pi})$. But, near $\Sigma = -\pi/2$:

$$\mathcal{J}(\beta; \Sigma; \chi) - \chi = (\Sigma + \pi/2) - \frac{1}{6}\chi^3 + \chi(\beta - 1) + o(|\Sigma + \tfrac{\pi}{2}| + |\chi(\beta-1)| + |\chi|^3) \tag{8.12}$$

In particular, for small ε, $\partial^3 G/\partial\chi^3 \neq 0$, for $|\chi| < A$. Further, for $|\beta - 1|, |\Sigma + \pi/2|$ small, $(\mathcal{J} - \chi)(\chi = A) > 0$, $(\mathcal{J} - \chi)(\chi = -A) < 0$, and the same is true for G(A),G(-A). By Rolle's theorem, it follows that there are at most three and at least one root of G on $|\chi|<A$. By changing $\beta - 1$, we can vary the number of roots from one to three and meet on the way at least one saddle - node bifurcation.

We can summarize this Section in:

<u>Theorem 8.1</u>: If $\beta<\beta_e(\Sigma) - \sigma$, $\sigma > 0$, the invariant sets of the (half period) Poincaré mapping $\mathbb{P}(\beta;\Sigma;\Lambda;\chi;\varepsilon)$ of Duffing's equation consist, for ε sufficiently small, of fixed points and periodic points of period two only - with the possible exception of a small neighbourhood, vanishing as $\varepsilon \to 0$, of $\chi = 0, \beta = 1, \Lambda = R_{oo}$, $\Sigma = -\pi/2$. These invariant points are in one - to - one correspondence with those of $\mathbb{J}(\beta;\Sigma;\chi)$, eqn. (7.13) and approach the latter as $\varepsilon \to 0$. The bifurcation lines $\beta_s(\Sigma;\varepsilon)$, $\beta_F(\Sigma;\varepsilon)$ approach those of $\mathbb{J}$ as $\varepsilon \to 0$. We recall, $\Delta < C\Gamma^{1/4}$.

This is, in fact, the main conclusion of this paper.

Clearly, we expect the transition to chaotic motion present in $\mathbb{J}(\beta;\Sigma)$ when β increases through β_c near $\Sigma = -\pi/2$, to occur also in $\mathbb{P}$, for small ε . This is, in fact, the contents of a general theorem of van Strien (Ref.[30]).

<u>Comment 8.4</u>: Theorem 8.1 establishes in particular the uniqueness of the periodic solutions of Duffing's equation for $C\Gamma^{1/4} > \Delta > \Delta_c(\Gamma) + \sigma$, for any $\sigma > 0$ (cf. Eqn. (1.2)), if Γ is only sufficiently large. Indeed, at no stage did we impose the restriction $k(\varepsilon) = O(1)$ as $\varepsilon \to 0$. If $k(\varepsilon) \to \infty$ as $\varepsilon \to 0, \beta \to 0$ and the limiting form of $\mathbb{P}$ is $\Lambda = R_{oo}$, $\chi = 0$. The upper bound on Δ is due to the boundary layer structure used in deriving uniform approximations to solutions of (1.6) (cf. Sect. I, eqns. (1.7)ff.).

IX. Uniqueness of periodic solutions at high Δ.

Although there is intuitively no doubt that uniqueness will persist even if $\Delta > C\Gamma^{1/4}$, we shall give a short proof of this below, for the sake of a clear conscience. The method of proof has been suggested by a referee of a previous (incomplete) version of this paper (Ref.[14]). In Ref.[14], uniqueness at high Δ was established in a more complicated manner, which also justified the validity of approximants like (3.8) to the periodic solutions in question. In the following, this latter topic is left out.

We assume first $C\,\Gamma^{1/4} < \Delta < B\,\Gamma^{2/3}$ and use the variable τ of (1.12). By Liapunov methods, we establish first a uniform bound on the solutions of (1.6) for t large enough. Let to this end $v(t) = x(t) - (\sin t)^{1/3}$, so that:

$$\varepsilon \ddot{v} + 2\mu\,\dot{v} + 3\,v\,(\sin t)^{2/3} + 3\,v^2\,(\sin t)^{1/3} + v^3 = h(t) \tag{9.1}$$

where $h(t) = O(\mu\,t^{-2/3})$, $\dot{h}(t) = O(\mu\,t^{-5/3})$, if $|t| > b\mu^{3/5}$.

<u>Lemma 9.1</u> : If $B\Gamma^{2/3} > \Delta > C\Gamma^{1/4}$, every solution x(t) of (1.6) obeys eventually: (B small enough)

$$|v(t)| < K\mu^{1/5}, \qquad |\dot{v}(t)| < K\mu^{-2/5} \tag{9.2}$$

for $t < -b\mu^{3/5}$ (mod π).

<u>Proof:</u> Assume first $\varepsilon/\mu^2 < D$ ($\Delta > \sqrt{D}\ \Gamma^{1/3}$). With $v(t) = \bar{v}(t)\exp[-C(t-t_o)/\mu]$ $C = k\mu^{\alpha}$, $0 \leq \alpha \leq 2/5$, and a Liapunov function similar to (2.11), we establish, for $t < -b\mu^{3\alpha/2}$ the differential inequality [32]:

$$\frac{dL_{\bar{v}}}{dt} < (2L_v)^{1/2}\ \ h(t)\,\exp\left[C(t - t_o)/\mu\right] \tag{9.3}$$

where we have used $|\ \varepsilon\dot{\bar{v}} + 2\mu(1 - C\,\varepsilon/\mu^2)\,\bar{v}\,| < (2L_v)^{1/2}$. Integrating (9.3) and using $L_{\bar{v}}(t_o) = O(\mu^2)$ (cf. Lemma 2.1), $L(t) \doteq L_{\bar{v}}\exp[-2C(t-t_o)/\mu]$, we obtain that $L(t)^{1/2} = O(\mu^{2-\alpha}/t^{2/3})$ after a time interval of $O(\mu^{\frac{3}{5}-\sigma})$, $\sigma > 0$, starting at $t_o < 0$. From:

$$\varepsilon\,\dot{v} + 2\mu(1 + C\,\varepsilon/\mu^2)\,v = O(\mu^{2-2\alpha}) \tag{9.4}$$

we deduce $v = O(\mu^{1-\alpha}/t^{2/3})$after a time interval of $O((\varepsilon/\mu)^{1-\sigma})$, $\sigma > 0$, for $t < -b\mu^{3\alpha/2}$. It follows that $\partial G/\partial v = O(\mu^{1-\alpha})$ and from the differential equation we obtain $\dot{v} = O(\mu^{-\alpha})$, for $t < -b\mu^{3\alpha/2}$. With $\alpha = 2/5$, we obtain the statement (9.2).

If $\mu^2/\varepsilon < B_1$, we proceed as above, with $v = \bar{v}\exp[-C\mu(t-t_o)/\varepsilon]$.

<u>Comment 9.1:</u> The proof above also establishes bounds on $|v(t)|$, $|\dot{v}(t)|$ on $-t_o < t < -b\mu^{3\alpha/2}$, by choosing $C = k\mu^{\alpha}$, $0 \leq \alpha \leq 2/5$.

<u>Lemma 9.2</u>: If $|\eta(-\tau_o)|$, $|d\eta/d\tau\,(-\tau_o)| < C$, then $|\eta(\tau)|$, $|d\eta/d\tau| < D$ for $\tau \in (-\tau_o + \delta\,,\,\tau_o)$; δ may be made arbitrarily small, for $\varepsilon/\mu^{8/5}$ small enough (η of (1.12)).

<u>Proof:</u> We use eqn. (1.13) and write $f(\tau;\mu)$ for the right hand side; $|f(\tau;\mu)|$ $<$ const for $|\tau| < \tau_o$. Consideration of: $(\nu \equiv \varepsilon\mu^{-8/5})$

$$L = \frac{1}{2}\left(\nu\frac{d\eta}{d\tau} + 2\eta\right)^2 + \frac{\nu}{4}\eta^4 \qquad (9.5)$$

shows that $L(\tau) < \mathrm{const}(\tau_o)$ for $|\tau| < \tau_o$. It follows that $|\eta| < D$, $|d\eta/d\tau| < D$ after a time interval of $O(\varepsilon/\mu^{8/5})$.

Lemma 9.3: If $B\Gamma^{2/3} > \Delta > C\Gamma^{1/4}$, every solution $x(t)$ of (1.6) which obeys $|x(\tau_o\mu^{3/5})| < D\mu^{1/5}$, $|\dot{x}(\tau_o\mu^{3/5})| < D\mu^{-2/5}$ obeys at $t = t_1$, $a < t_1 < \pi - a$:

$$|v(t)| < K\mu \quad ,|\dot{v}(t)| < K \qquad (9.6)$$

The proof is the same as for Lemma 9.1.

Comment 9.2: According to Lemmas 2.1, 9.1-3, the rectangle $\tilde{D}: |v(t)| < C_1$, $|\dot{v}(t)| < C_2\,\varepsilon^{-1/2}$ at $t = -\pi/2$ is mapped by the solutions of (1.6) into the rectangle (9.6) at $t = \pi/2$, which is strictly contained in $-\tilde{D}$. Thus, the half period Poincaré map has at least one fixed point, given by an odd periodic solution $x_o(t)$.

Consider now the difference between two solutions x_1, x_o of (1.6): $u = x_1 - x_o$:

$$\varepsilon\ddot{u} + 2\mu\dot{u} + 3w^2 u = 0 \qquad (9.7)$$

with $w = (x_1^2 + x_1x_o + x_o^2)/3$.

Lemma 9.4: Let:

$$L(u, \dot{u}, t) = \frac{(\varepsilon\dot{u} + 2\mu u(1 - \varepsilon\alpha/2\mu))^2}{2} + \frac{u^2}{2}(3w^2 + \varepsilon\alpha^2 - 2\mu\alpha) \qquad (9.8)$$

If $\alpha = C\mu^{-3/5}$, $L(u,\dot{u},t)$ is strictly positive definite in u, $\dot{u}$ for $t < -d\mu^{3/5}$ and, if u, $\dot{u}$ are solutions of (9.7):

$$L(u,\dot{u},t) < L(t_o)\exp[-2\alpha(t - t_o)] . \qquad (9.9)$$

for $t < -d\mu^{3/5}$:

Proof: We write $u = \bar{u}\exp[-\alpha(t - t_o)]$. Then, $L(u,\dot{u},t) \equiv \bar{L}(\bar{u},\dot{\bar{u}},t)\exp[-2\alpha(t-t_o)]$. From the differential equation for $\bar{u}$, we verify $d\bar{L}/dt < 0$ if $t < -d\mu^{3/5}$. To this end, we use $|w^2| > C\mu^{2/5}$, $|w\frac{dw}{dt}| < C/\mu^{1/5}$ if $t < -d\mu^{3/5}$, as shown in Lemma 9.1. This ends the proof.

From (9.8),(9.9), we deduce:

$$|\varepsilon\dot{u} + 2\mu u(1 - \varepsilon\alpha/2\mu)| < (2L(t_o))^{1/2}\exp[-\alpha(t - t_o)] \qquad (9.10)$$

From (9.10), we conclude that, after a time $\varepsilon/\mu < 1/\alpha$, starting at t_o:

$$|u(t)| < C\frac{\sqrt{2L(t_o)}}{\mu}\exp[-\alpha(t - t_o)] \qquad (9.11)$$

for $t < -d\mu^{3/5}$. We can use the differential equation (9.7) to establish a bound on $|\dot{u}(t)|$, using (9.11):

$$|\dot{u}(t)| < C\frac{(2L(t_o))^{1/2}}{\mu^2}w^2\exp[-\alpha(t - t_o)] \qquad (9.12)$$

Since, by Lemma 2.1, $L(t_o) = O(\varepsilon + \mu^2)$, we see that $|u(t)|$, $|du/dt|$ may be made as small as one wishes at $t \sim - d\mu^{3/5}$, provided $\mu < B$ is small enough.

Lemma 9.5: $|u(\tau_o)|, |du/d\tau\,(\tau_o)| < \text{const}\,|u(-\tau_o)|, |du/d\tau\,(-\tau_o)|$ (9.13)

The proof is obtained from Gronwall's inequality applied to:

$$u(\tau) = A + B\exp\left[-\frac{2}{\nu}(\tau+\tau_o)\right] - \int_{\tau_o}^{\tau} \frac{e^{-2(\tau-\tau')/\nu} - 1}{2}\, 3W^2(\tau')\,u(\tau')d\tau' \qquad (9.14)$$

where $\nu = \varepsilon\,\mu^{-8/5}$, $B = O(\nu)$, $W = w\mu^{-1/5}$.

ting the reasoning of Lemma 9.4 for $t > 0$, we deduce that $L(\pi/2) < C(\mu)L(-\pi/2)$, with $C(\mu) \to 0$ as $\mu \to 0$. Now, from Lemma 9.1, $w(t) = (\sin t)^{1/3} + O(\mu)$, (cf. Comment 9.1) so that $C_1L(-\pi/2) < L(\pi/2) < C_2L(-\pi/2)$, with $C_1, C_2 \sim 1$. It follows that the half period Poincaré map is, for μ small enough, a contraction in the norm $L(u,\dot{u},-\pi/2)$. Thus:

Theorem 9.1: If $B\Gamma^{2/3} > \Delta > D^{-1}\Gamma^{1/4}$, B, D sufficiently small, eqn. (1.6) admits of a unique (odd) periodic solution $x_o(t)$ for Γ large enough.

We turn next to the regime $\Delta > B\Gamma^{2/3}$; we use eqns. (1.8),(1.9), with the variables (1.7). Qualitatively, if $\Delta < B\Gamma^{2/3}$, with B small, then a motion starting with $x(0) = a \neq 0$, $\dot{x}(0) = 0$ gets close to $(\sin t)^{1/3}$ within a half period of the external force (if Γ is large enough, cf. Lemma 9.1). If Δ is larger, then :(a) the limiting motion is no longer close to $(\sin t)^{1/3}$ and (b) the approach to the periodic solution may extend over many 2π periods.

We shall only sketch the proof of uniqueness, since it is essentially the same as that of Theorems 3.1, 3.2. Details are given in Refs.[14,31]. First, we have:

Lemma 9.6: If $\Delta > B\Gamma^{2/3}$, eqn. (1.8) admits of a (2π - odd) periodic solution $z_p(t)$, which departs from

$$z_a^{(n)}(t) = z_o(t) + \bar{\varepsilon}\,z_1(t) + \bar{\varepsilon}^2 z_2(t) + \ldots + \bar{\varepsilon}^n\,z_n(t) \qquad (9.15)$$

uniformly by $O(\bar{\mu}\,\bar{\varepsilon}^{n+1})$; $z_o(t)$ is the unique periodic solution of (1.9) and $z_i(t)$ are obtained by formal iteration of (1.8).

The proof is done by Newton's method, noting that $\mathcal{Y}(z_a^{(n)}(t)) = O(\bar{\mu}\,\bar{\varepsilon}^{n+1})$ (cf. (3.10)) and using two solutions of the variational equation to (1.8) around $z_a^{(n)}(t)$. Estimates of these solutions are obtained by WKB methods (see Refs.[14,31]).

For uniqueness, we show:

Lemma 9.7: The difference $u(t) = z(t) - z_p(t)$ obeys $|u(t)| < C \exp[-\lambda(t-t_o)]$ with $\lambda = k\mu$, k sufficiently small, provided $|u(t_o)|$, $|\dot{u}(t_o)|$ are small enough.

This is proved by transforming to an integral equation and investigating the conditions for contraction.

To ensure that t_o exists, with the properties of Lemma 9.7, we prove, by Liapunov function methods:

Lemma 9.8: There exists a time t_o, so that $|u(t_o)| = O(\bar{\varepsilon})$.

It turns out (see Refs.[14,31]), that this is sufficient to state:

Theorem 9.2: Eqn. (1.6) admits of a unique solution if $\Delta > B\Gamma^{2/3}$, for Γ large enough.

This closes the discussion of the uniqueness problem at high Γ.

X. Summary

The above establishes the existence of an infinite sequence of bifurcation curves alternatively of the saddle - node and odd - 2π simply periodic type for large $\Gamma, \Delta \sim k \ln \Gamma$ in the $\Gamma - \Delta$ plane of the Duffing equation (1.1). With Theorem 8.1, this is even the complete picture of bifurcations and invariant sets of the (half period) Poincaré map if the damping is large enough. If $\Delta > C\Gamma^{1/4}$, the situation is simple and described in Theorems 9.1, 9.2. The exposition has the drawback that much detail is present, part of which is inessential to the appearance of bifurcations and chaos.

Indeed, some of the "kinematical" complications, with two special solutions X_L, X_R of eqn. (1.6) (cf. Sect.III), boundary layers, etc. may be seen in pure form in the linear equation:

$$\varepsilon \ddot{x} + 2\mu\dot{x} + x = (\sin t)^{1/3} \qquad (10.1)$$

Eqn. (10.1) has only one periodic solution for all ε, μ. If $|t|>0$, it is close to the outer expansion associated to (10.1), for small ε, μ. This expansion is obtained by iterating (10.1):

$$x_{out} = (\sin t)^{1/3} - (2\mu/3)(\sin t)^{-2/3} \cos t + O(\varepsilon + \mu^2) \qquad (10.2)$$

The terms of the expansion diverge as $t \to 0$; for small $|t|$, we match it to a boundary layer correction. Changing variables in (10.1) to:

$$x = \varepsilon^{1/6} X \qquad , \quad t = \varepsilon^{1/2} \tau \tag{10.3}$$

we obtain:

$$\frac{d^2X}{d\tau^2} + 2 \frac{\mu}{\sqrt{\varepsilon}} \frac{dX}{d\tau} + X = \frac{(\sin \varepsilon^{1/2}\tau)^{1/3}}{\varepsilon^{1/6}} = \tau^{1/3} - \varepsilon \frac{\tau^{7/3}}{18} + O(\varepsilon^2). \tag{10.4}$$

From (10.4), we can derive a left and a right hand inner expansion: the left hand expansion is made up of solutions of (10.4) (decomposed into coefficients of powers of ε) behaving e.g. like $X_{Lo} \sim \tau^{1/3}$ as $\tau \to -\infty$, etc.; the right hand expansion is obtained as in eqns. (3.21) ff., Sect. III. These expansions are improved to left and right hand solutions x_L, x_R of (10.4) starting from approximants, e.g.:

$$x_{aL}(t) = x_o^{(K,L)} \chi_o(t) + \varepsilon^{1/6} X_i^{(Q)} \chi_i(t) \tag{10.5}$$

as in (3.8). The differences:

$$\Delta X = \varepsilon^{-1/6}(x_L(0) - x_R(0)), \qquad \Delta X' = \varepsilon^{1/3}\left(\frac{dx_L}{dt}(0) - \frac{dx_R}{dt}(0)\right) \tag{10.6}$$

have finite limits, not both vanishing, as $\varepsilon \to 0$. The periodic solution $x_p(t)$ of (10.1) is obtained by superimposing on x_L, x_R suitable damped oscillations, i.e. solutions of:

$$\varepsilon \ddot{v} + 2\mu \dot{v} + v = 0 \tag{10.7}$$

We call $v_L(t)$, $v_R(t)$ the solutions of (10.7) which are such that:

$$x_p(t) = x_L(t) + v_L(t), \quad t < 0 \quad , \quad x_p(t) = x_R(t) + v_R(t) \ , \ t > 0 \tag{10.8}$$

To obtain the magnitude of $v_L(t)$, $v_R(t)$, we notice that, if $|v_L| < \varepsilon^{1/6}$ at $t = 0$, then $v_R(\pi/2) \sim \varepsilon^{1/6}\ \varepsilon^{k\pi/2}$, if $\mu = k\,\varepsilon \ln 1/\varepsilon$, $k = O(1)$; also, $(dv_R/d\tau)(\pi/2) \sim \varepsilon^{1/6}\varepsilon^{k\pi/2}$. **Thus, the half period Poincaré map sends domains of size ε^s, $s > 1/6$ at $t = -\pi/2$ into a domain of size $\varepsilon^{s+k\pi}$ around a point with coordinates $(v_R, dv_R/d\tau) \sim \varepsilon^{1/6+k\pi/2}$ at $t = \pi/2$. In particular, the fixed point has coordinates $v_R, dv_R/d\tau \sim \varepsilon^{1/6+k\pi/2}$**

For small ε, μ, the uniqueness of the fixed point may also be interpreted as an (asymptotic) "loss of memory" of the (half period) Poincaré map about the initial phase Ψ_o in a disk $|v_R|^2 + |dv_R/d\tau|^2 < C(\varepsilon^{1/6 + k\pi/2})^2$ at $t = -\pi/2$ (cf. eqn. (4.3)). Indeed, the initial conditions for $v_R(t)$, eqn. (10.8) at $t = 0$ are : $v_R(0) = \varepsilon^{1/6}(\Delta X + O(\varepsilon^{k\pi}))$, $dv_R/d\tau\,(0) = \varepsilon^{1/6}(\Delta X' + O(\varepsilon^{k\pi}))$ and the knowledge about Ψ_o is contained in the $O(\varepsilon^{k\pi})$ terms. Now, the phase Ψ of the

solution at $t = \pi/2$ (measured in a disk of radius $\varepsilon^{1/6 + k\pi/2}$) is stable against small displacements of the initial values at $t = 0$. Thus, its sensitivity to Ψ_o vanishes as $\varepsilon \to 0$ and the coordinates of the fixed point are fixed merely by ΔX, $\Delta X'$. (eqn. (10.6))

This is precisely what happens also for the Duffing equation if the damping is high enough. However, the stability of the phase at $t = \pi/2$ with respect to small changes of the initial conditions at $t = 0$ ceases to hold if the damping is small enough. To describe the mechanism through which this occurs, consider the nonlinear analogue of (10.7) appearing in the Duffing equation: ($\theta = \theta_R$ of eqn. (4.2)):

$$\frac{d^2\tilde{w}}{d\theta^2} + 2\frac{\gamma\varepsilon^{1/8}}{X_R(t)}\frac{d\tilde{w}}{d\theta} + \tilde{w}(1 + \tilde{g}) + \tilde{w}^2\tilde{k} + \tilde{w}^3\frac{\tilde{k}^2}{3} = 0 \qquad (10.9)$$

with γ of (1.15), $\tilde{w} = w\,\varepsilon^{kt}$, $\tilde{k} = k\,\varepsilon^{-kt}$ of (6.4),(6.5), $\tilde{g}$ is a correction to g of (6.4). If $\gamma = \varepsilon = 0$, the (almost) harmonic oscillation described by the linear part of (10.9) acquires an additional phase $\text{const}\cdot R^2\theta^{1/4}$, where R is the amplitude of the oscillation. In the variable θ, $\tilde{k} \sim \theta^{-3/8}$; although negligible for large θ, this perturbation is not integrable and leads to sizable effects ($\sim \theta^{1/4}$) for large θ. If γ, ε are finite (but $\gamma \to 0$ as $\varepsilon \to 0$), the amplitude R decreases with time like $e^{-\gamma\tau}$, with $\tau = t\varepsilon^{-3/8}$; it turns out that the additional phase no longer increases indefinitely, but rather levels off at a value $\gamma^{-1/3}$ when $\tau \sim \gamma^{-1}$; it is still roughly proportional to the square of the amplitude R_o at some finite value of θ.

Now, if $\gamma = 0$, no matter how small the extension of the disk of initial conditions in the (u_R, $du_R/d\tau$) plane at $\tau = 0$ (the notation of (6.1)), it generates trajectories of (10.9) that, although possessing almost the same amplitude, reach macroscopically different phases, if we only wait long enough in θ. If $\varepsilon \neq 0$ ($\gamma \neq 0$), the extension of the image at $t = 0$ of the disk D with radius $\varepsilon^{3/16}\,\varepsilon^{k\pi/2}$ in the v_L, $\dot{v}_L$, $dv_L/d\theta$ plane (cf. eqn. (4.3)) of initial conditions at $t = -\pi/2$ is itself $\sim \varepsilon^{k\pi}$.

It follows that changes within the disk D obtained by varying the phase Ψ_o may lead to sizable effects of the phase at $t = \pi/2$ (in the disk of radius $\varepsilon^{3/16}\,\varepsilon^{k\pi/2}$) if:

$$\frac{\varepsilon^{k\pi}}{\gamma^{1/3}} \sim 1 \tag{10.10}$$

If this is true, we expect that, even as $\varepsilon \to 0$, the Poincaré map no longer loses memory of the initial phase at $t = -\pi/2$, but acquires a nontrivial form. This is the mapping $\overline{\mathcal{J}}$, eqn. (7.13), which does display bifurcations and chaotic behaviour.

Finally, we indicate roughly why the inclusion of a linear term ky in eqn. (1.1) is of no importance for the pattern of bifurcations at high Γ and Δ, provided k is independent of both. First, the change of variables (1.5) leads to:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + x^3 + \varepsilon k x = \sin t \tag{10.11}$$

In the outer expansion associated to (10.11), the terms ~ 1 and $\sim \mu$ are unchanged if $\varepsilon/\mu \to 0$ as $\varepsilon \to 0$. In the inner expansion with the variables (1.15), the harmonic term does not occur in the leading equation (3.5) but only modifies (3.6) to:

$$\frac{d^2\eta_1}{d\tau^2} + 2\gamma \frac{d\eta_1}{d\tau} + (3\eta_o^2 + k)\eta_1 = -\frac{\tau^3}{6} \tag{10.12}$$

Since $\eta \sim \tau^{1/3}$, the leading terms in the asymptotic behaviour of the solution are not affected by k. Thus, X_R, X_L are modified with respect to the k = 0 situation only through terms whose relative weight vanishes as $\varepsilon \to 0$, if the matching in (3.8) is performed at $t \sim \varepsilon^{\alpha}$, $0 < \alpha < 3/8$.

The equations for v_L, v_R (cf. (4.1), (6.18)), which contain the nonlinear effects responsible for bifurcations are changed , e.g. to:

$$\varepsilon \ddot{v}_R + 2\mu \dot{v}_R + (3 X_R^2 + \varepsilon k) v_R + 3 X_R v_R^2 + v_R^3 = 0 \tag{10.13}$$

In the versions (4.8), (6.4) of these equations, the harmonic termleads to the addition to the function $g(\theta)$ in the linear part of terms of $O(\varepsilon^{3/4}\theta^{-1/2})$, whose integrals over θ- intervals of $O(\varepsilon^{-1/2})$ vanish as $\varepsilon \to 0$. Thus, in the limit $\varepsilon \to 0$, these terms do not lead to any change in the phase of the oscillations and therefore in the asymptotic form (7.13) of the Poincaré mapping. In particular, the sign of k is irrelevant at large Γ .

A generalization of the results of this paper to Duffing - like equations (obtained by replacing y^3 by y^{2n+1} in (1.1)) and to other forms of the forcing term is under consideration.

Appendix A: Proof of Lemma 3.2.

We consider first (3.5) and write:

$$\eta_o(\tau) = \tau^{1/3}\sum b_{klo}\,\tau^{-5k/3 - 8l/3}\gamma^k + u(t) \equiv \eta_{o,K,L}(\tau) + u \qquad (A.1)$$

where the b_{klo} are determined by substitution in (3.5) and equating coefficients of equal powers of $\gamma\tau^{-5/3}$ and $\tau^{-8/3}$. The result is a nonlinear equation for $u(\tau)$:

$$\frac{d^2u}{d\tau^2} + 2\gamma\frac{du}{d\tau} + 3\eta_{oKL}^2\,u + 3\eta_{oKL}\,u^2 + u^3 = O(\tau^{1/3 - 5(K+1)/3 - 8(L+1)/3}\gamma^{K+1}) \qquad (A.2)$$

As $\tau\to-\infty$, the solutions of the linear homogeneous part of (A.2) behave like $u_{1,2} \simeq \exp(-\gamma\tau)\binom{\sin}{\cos}(\tau^{4/3})$; using the variation of the parameters, it is straightforward to show by a contraction argument that (A.2) admits of a solution which falls off at infinity like $\tau^{1/3 - 5K/3 - 8L/3}\gamma^K$. If $\gamma\neq 0$, the same argument shows that the solution is unique in the class of functions $|u(\tau)| < \tau^q$, $q>0$ as $\tau\to-\infty$. This establishes the existence and uniqueness of $\eta_o(\tau)$ and also gives its asymptotic expansion.

The equations for $\eta_q(\tau), q>0$ are linear and the argument is even simpler.

The task is to show that the coefficients b_{klq} are the same as the a_{klq} of (3.3). We show that the equations determining them are the same. Consider b_{klo} first; writing $\gamma\tau^{-5/3} = x$, $\tau^{-8/3} = y$, the equations for the b_{klo} read:

$$\sum_{k,l\geq 0} b_{klo}\left(-\frac{2}{3} - \frac{5k}{3} - \frac{8l}{3}\right)\left(\frac{1}{3} - \frac{5k}{3} - \frac{8l}{3}\right)x^k y^{l+1} + \sum_{k,l\geq 0} b_{klo}\left(\frac{1}{3} - \frac{5k}{3} - \frac{8l}{3}\right)x^{k+1}y^l$$
$$+ \left(1 + \sum_{k,l\neq(0,0)} b_{klo}\,x^k y^l\right)^3 = 1 \qquad (A.3)$$

and it is easy to see that b_{klo} may be expressed recurrently and uniquely in terms of $b_{k'l'o}$ with $k'<k, l'\leq l$ or $k'\leq k, l'<l$; $b_{ooo} = 1$.

Now, the equations for the a_{klq} are obtained by substituting (3.1),(3.3) into (1.6) and equating coefficients of equal powers k,l,q of $x = \mu t^{-5/3}$, $y = \varepsilon t^{-8/3}$ and $z = t^2$ in turn. As is expected from (1.15), the set of equations for a_{klo} is decoupled from those with higher q and is given implicitly by the same expression (A.3). If $q>0$, the a_{klq} (or b_{klq}) are obtained recurrently in terms of $a_{k'l'q'}$ with $k'<k$, $l'\leq l$, $q'\leq q$ and permutations.

Appendix B: Proof of Lemma 3.4.

The solutions $w_{1,2}(\tau;\varepsilon)$ depend on ε through Φ and the lower limit of integration in (3.18). At $\varepsilon = 0$, $\Phi(\tau;\varepsilon=0)$ is given for $\tau<0$ by:

$$\Phi(\tau;\varepsilon=0) = 3\eta_{oo}^2(\tau) \tag{B.1}$$

with $\eta_{oo}(\tau)$ of (3.22), $\eta_{oo}(\tau)\sim\tau^{1/3} + o(\tau^{-1/2})$ as $\tau\to-\infty$. It is true that, uniformly on $I_\tau = [-\varepsilon^{-3/8+a},0]$, $a>0$:

$$\lim\,(1+\tau^{2/3})^{-1}\Phi(\tau;\varepsilon) = (1+\tau^{2/3})^{-1}\Phi(\tau;0) \tag{B.2}$$

Indeed, on an interval $\bar{I}_\tau = I_\tau\setminus[-\tau_0,0]$, we have the uniform estimate: (cf. (3.8)):

$$\eta_{iL}^{(Q)}(\tau)\equiv x_{iL}^{(Q)}(\tau)\,\varepsilon^{-1/8} = \eta_{oo}(\tau)(1+O(\gamma/\tau^{4/3},\ \varepsilon^{3/4}\tau^2)) \tag{B.3}$$

obtained from the asymptotic expansion (3.7); this implies (B.2). Now, (B.2) implies, with (3.19):

$$\Phi^{-1/2}(\tau;\varepsilon)\,R(\Phi) < C/\tau^3 \tag{B.4}$$

independently of ε, on $\bar{I}_\tau$.

Now, for $|\tau|>\varepsilon^{-3/8+a}$, using (3.1), (3.3), $x_{aL}(t)$ and its first two derivatives are bounded from above and below by $c_u t^{1/3}$, $c_d t^{1/3}$ and their derivatives, $c_u > c_d > 0$. Thus, (B.4) holds all the way down to $-(\pi/2)\,\varepsilon^{-3/8}$. With this, Gronwall's inequality leads in (3.18) to:

$$|w_i(\tau;\varepsilon) - w_i^{(w)}(\tau;\varepsilon)| < C\tau^{-13/6}, \quad |dw_i/d\tau - dw_i^{(w)}/d\tau\,(\tau;\varepsilon)| < C\tau^{-11/6} \tag{B.5}$$

for $-\pi/2\ \varepsilon^{-3/8} < \tau < -\tau_o$, independently of ε.

Now, $w_i^{(w)}(\tau;\varepsilon)$ has a finite limit $w_i^{(w)}(\tau;0)$ as $\varepsilon\to 0$, at any finite τ. We can then let formally $\varepsilon\to 0$ in (3.18) and obtain:

$$w_{1,2}(\tau;0) = w_{1,2}^{(w)}(\tau;0) + \frac{1}{\Phi(\tau;0)}\int_{-\infty}^{\tau} R(\Phi(0))\,\Phi^{-1/4}(\tau;0)\,\sin\Big[\int_{\tau'}^{\tau}\Phi^{1/2}(\tau'';0)\,d\tau''\Big]w_{1,2}(\tau')\,d\tau' \tag{B.6}$$

Eqn. (B.6) has a unique solution $w_{1,2}(\tau;0)$ for $\tau <$ const, as follows from (B.4) and a contraction argument.

We subtract now (B.6) from (3.18), separate out the integral over $[-\varepsilon^{-3/8+a},-\tau_o]$, bound those on the remaining segments using (B.4) and establish by means of Gronwall's inequality and of (B.5) that $w_1(\tau;\varepsilon)$ converges to $w_1(\tau;0)$ uniformly on $-[\varepsilon^{-3/8+a},-\tau_o]$, provided a is chosen appropriately large. The condition on a is due to the requirement that the phases of $w_1^{(w)}(\tau;\varepsilon)$, $w_1^{(w)}(\tau;0)$ approach each other on $-[\varepsilon^{-3/8+a},-\tau_o]$. The condition for this to happen is $\varepsilon^{3/4}\tau^{10/3}\to 0$, which means $a > 3/20$. With a further restriction on a, we may even ensure that $dw_1/d\tau(\tau;\varepsilon)$ approaches $dw_1/d\tau(\tau;0)$ on $[-\varepsilon^{-3/8+a}, -\tau_o]$.

Using (B.2) and the fact that $w_1(\tau;\varepsilon)$, $w_1(\tau;0)$ are $\varepsilon^p (p>0)$ close at $\tau = -\tau_o$, we deduce by comparing the equations

$$\frac{d^2w}{d\tau^2} + \Phi(\tau;\varepsilon)w = 0 \tag{B.6}$$

for ε and $\varepsilon = 0$, that $w_1(\tau;\varepsilon)$, $w_1(\tau;0)$ approach each other uniformly on $[-\tau_o, 0]$ as $\varepsilon \to 0$ (as well as their derivatives)

A coarse estimate through the above steps gives:

$$|w(\tau;\varepsilon) - w(\tau;0)| \quad , \left|\frac{dw}{d\tau}(\tau;\varepsilon) - \frac{dw}{d\tau}(\tau;0)\right| \quad = O(\varepsilon^{15/64}) \tag{B.7}$$

for τ on $[-\tau_o, 0]$.

Appendix C: Proof of Lemma 4.6

The object of interest is the energy associated to (4.30):

$$E(\theta) = \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 + \frac{1}{2}w^2(1+g) + \frac{1}{3}w^3 h_1(\theta) + \frac{1}{12}w^4 h_1^2(\theta) \tag{C.1}$$

Now, for small ε :

$$\frac{d}{d\theta}h_1(\theta) < 0 \tag{C.2}$$

if $t \in [T_{oR}\varepsilon^{3p}, \varepsilon^{3p-\delta}]$, since $\varepsilon^{kt} \simeq 1$ and $X_R(t)$ is monotonically increasing. This means:

$$\frac{dE}{d\theta} < \frac{w^2}{2}\left|\frac{dg}{d\theta}\right| + \frac{|w^3|}{3}\frac{dh_1}{d\theta} \tag{C.3}$$

Letting:

$$\theta_a = \theta(T_{oR}\varepsilon^{3p}) \tag{C.4}$$

using the facts that $\int_{\theta_a}^{\theta} |dg/d\theta|\, d\theta' = \tilde{g}(\theta) < \text{const}$ and $w^2 < C_o E$, multiplying (C.3) by $\exp[-C_o\,\tilde{g}(\theta)]$ and integrating between $\theta_1, \theta_2 > \theta_a$:

$$E(\theta_2)\exp[-C_o(\tilde{g}(\theta_2) - \tilde{g}(\theta_1))] < E(\theta_1) + \int_{\theta_1}^{\theta_2} \frac{1}{3}|w|^3\left|\frac{dh_1}{d\theta}\right|\exp[-C_o(\tilde{g}(\theta') - \tilde{g}(\theta_1))]\, d\theta' \tag{C.5}$$

Now, it is true that:

$$\frac{C_-}{\theta}\,\frac{1}{T_{oR}^{1/2}}\left(\frac{\theta_a}{\theta}\right)^{3/8} < \left|\frac{dh_1}{d\theta}\right| < \frac{C_+}{\theta}\,\frac{1}{T_{oR}^{1/2}}\left(\frac{\theta_a}{\theta}\right)^{3/8} \quad , C_-, C_+ \sim 1 \tag{C.6}$$

so that the right hand side of (C.5) is bounded by a constant for all θ , if:

$$|w(\theta)| < C(\theta/\theta_a)^{1/8-q} \tag{C.7}$$

for some $q > 0$. We show in fact first the following:

Lemma C.1: Assume:

$$E(\theta) = m_o^2(\theta)(\theta/\theta_a)^{3/4} \tag{C.8}$$

with $m_o(\theta) \to 0$ as $\theta \to \infty$. Assume further that, if $\theta_1 < \theta < \theta_1 + T$, $m_o(\theta) < Cm_o(\theta_1)$, with C independent of θ_1. Then $E(\theta) <$const for all θ.

Proof: Let $\tilde{E}(\theta) = E(\theta) \exp[\pm C_o \tilde{g}(\theta)]$. Since $\tilde{g}(\theta) \sim 1/\theta^2$ (cf.(4.9)), $\tilde{E}(\theta)$ obeys the same conditions as $E(\theta)$, with $m_o(\theta)$ replaced by $m(\theta)$. From (C.1), (C.8), we deduce :

$$|w(\theta)| < C_1\ m(\theta)\ (\theta/\theta_a)^{3/8} \tag{C.9}$$

for some $C_1 > 0$. Roughly speaking, the argument is as follows: as a consequence of (C.9) and the fact that $h_1(\theta) \sim (\theta_a/\theta)^{3/8}$, the cubic and quartic terms in (C.1) become negligible at high θ with respect to the quadratic terms. Thus, if the energy becomes unbounded as $\theta \to \infty$, its increase must be due to the quadratic terms; this,however, contradicts (C.5). To make this precise, consider a pair of points $\theta_2 = \theta_1 + T$, $\theta_1 > \theta_a$. Then (C.5),(C.9) imply:

$$m^2(\theta_2)(\theta_2/\theta_a)^{3/4} - m^2(\theta_1)(\theta_1/\theta_a)^{3/4} < D\, m^3(\theta_1)(\theta_a/\theta_1)^{1/4} \tag{C.10}$$

Multiplying by $(\theta_a/\theta_2)^{3/4}$ and using:

$$(1 + T/\theta_1)^{-3/4} \leqslant 1 - (3/4 - s)T/\theta_1 \tag{C.11}$$

where $s \to 0$ as $\theta_1 \to \infty$, we obtain

$$m^2(\theta_2) < m^2(\theta_1)[1 - (3/4 - \sigma)T/\theta_1] \tag{C.12}$$

where $\sigma \to 0$ if $\theta_1 \to \infty$. In (C.12) we have used the fact that $m(\theta) \to 0$ as $\theta \to \infty$. Iterating (C.12) for a sequence of points $\{\theta_n\}$, $\theta_n = \theta_{n-1} + T$, we obtain:

$$m(\theta_k) < \text{const}\ \frac{m(\theta_1)}{n^{3/8 + \sigma_1}} \tag{C.13}$$

with $\sigma_1 \to 0$ if $\theta_1 \to \infty$. With (C.8),(C.9), we see that $|w(\theta)| < C\,\theta^{\sigma_1}$, which complies with (C.7) if θ_1 is large enough. This ends the proof.

Comment C.1: Eqn.(4.30) is dependent on ε essentially through the interval in θ over which its solutions are considered. This interval increases indefinitely as $\varepsilon \to 0$. The ε dependence of $h_1(\theta), g(\theta)$ is negligible for small ε and

and $\theta < \varepsilon^{-1/2+\sigma}$, $\sigma>0$. The bound (C.8) is assumed to be independent of ε and to hold on $\theta_1 < \theta < \varepsilon^{-1/2+\sigma}$.

According to Lemma C.1, we are left with the obviously simpler problem of justifying (C.8). To this end, we let:

$$W = w(\theta_a/\theta)^{3/8} \tag{C.14}$$

which leads to:

$$\frac{d^2W}{d\theta^2} + \frac{3}{4}\frac{1}{\theta}\frac{dW}{d\theta} + W(1 + K(\theta)) + W^2 \tilde{h}_1(\theta) + W^3 \frac{\tilde{h}_1^2(\theta)}{3} = 0 \tag{C.15}$$

where:

$$K(\theta) = g - \frac{15}{64}\frac{1}{\theta^2} \qquad , \tilde{h}_1 = h_1(\theta/\theta_a)^{3/8} \tag{C.16}$$

It is true that:

$$\frac{dK}{d\theta} = O(\theta^{-3}) \ , \ \frac{d\tilde{h}_1}{d\theta} = O(\uparrow\theta^{-1/4} + \varepsilon^{3/4}\theta^{1/2}), \ \tilde{h}_1(\theta) \sim T_{oR}^{-1/2} \tag{C.17}$$

We shall show that the energy $L(\theta)$ associated to (C.15):

$$L(\theta) = \frac{1}{2}\left(\frac{dW}{d\theta}\right)^2 + \frac{W^2}{2}(1 + K) + \frac{W^3}{3}\tilde{h}_1 + \frac{W^4}{12}\tilde{h}_1^2 \tag{C.18}$$

has the property $L(\theta) \to 0$ as $\theta \to \infty$.

<u>Comment C.2</u>: If $L(\theta) \to 0$, $|W(\theta)|, |dW/d\theta|$ also vanish as $\theta \to \infty$. Then:

$$E(\theta) = \left(\frac{\theta}{\theta_a}\right)^{3/4} [L(\theta) + \frac{3}{4}\frac{dW}{d\theta}\frac{W}{\theta} + \frac{3}{8}\frac{W^2}{\theta^2}] \equiv \left(\frac{\theta}{\theta_a}\right)^{3/4} m_o(\theta) \tag{C.19}$$

Clearly, $L(\theta)(1 - C_1/\theta) < m_o(\theta) < L(\theta)(1 + C_1/\theta)$ and thus $m_o(\theta + sT) < Cm_o(\theta)$ if $L(\theta + sT) < CL(\theta)$, $0 < s < 1$.

Concerning $L(\theta)$, we state first:

<u>Lemma C.2</u>: $L(\theta)$ is bounded on $[T_{oR}\,\varepsilon^{3p}, \varepsilon^{3p-\delta}]$ by a constant L_o, independent of ε.

<u>Proof</u>: As a consequence of (C.18), $W^2, |W^3|, W^4 <$ const·L, where the constant is independent of ε. Integration of:

$$\frac{dL}{d\theta} < \text{const}\cdot L\left(\left|\frac{dK}{d\theta}\right| + \left|\frac{d\tilde{h}_1}{d\theta}\right| + \left|\frac{d}{d\theta}(\tilde{h}_1^2)\right|\right) \tag{C.20}$$

leads then to the statement of the Lemma.

<u>Comment C.3</u>: Integration of (C.20) between θ and $\theta + sT$ leads to: $(0 < s < 1)$

$$L(\theta + sT) < C\,L(\theta) \tag{C.21}$$

with a constant C close to unity. With Lemma C.1 and Comment C.2, the proof of Lemma 4.6 is completed if we show that $L(\theta) \to 0$ as $\theta \to \infty$ (cf. Comment C.1).

To establish this, we describe in more detail the motion (C.15) by comparing it with the motion $W^{(o)}(\theta)$ in the potential of (C.15), however at a fixed $\theta = \theta_1$ and without damping:

$$\frac{d^2W^{(o)}}{d\theta^2} + W^{(o)}(1 + K(\theta_1)) + W^{(o)2}\tilde{h}_1(\theta_1) + \frac{1}{3}W^{(o)3}\tilde{h}_1^2(\theta_1) = 0 \tag{C.22}$$

Concerning the latter, we have the following, which is straightforward:

Lemma C.3: (i) The motions $W^{(o)}(\theta)$ with energy bounded by L_o are periodic, with periods bounded from above (by C_u) and from below(by $C_d(L_o)$), independently of ε ;(ii) given $2 > k > 0$, there exists $k_1 < 1$, independent of the initial conditions, so that, if the motion $W^{(o)}(\theta)$ has period T_o, the θ-interval T_{ko} for which $(dW^{(o)}/d\theta)^2 > kL^{(o)}$ obeys $T_{ko} > k_1 T_o$.

We compare next the solutions of (C.15) to those of (C.22), over a time interval pC_u, p large, using Gronwall's inequality. The calculations are easy; details are given in Ref.[15]:

Lemma C.4: Assume $W(\theta)$, $W^{(o)}(\theta)$ obey the same initial conditions at $\theta = \theta_1$; the energy of $W^{(o)}(\theta)$ is L. Then, over a time interval of length pC_u:

(i) $$\max\left[|W(\theta) - W^{(o)}(\theta)|\ ,\left|\left(\frac{dW}{d\theta} - \frac{dW^{(o)}}{d\theta}\right)(\theta)\right|\right] < \frac{C(p,L_o)}{\theta_1} \tag{C.23}$$

(ii) if:

$$L \geqslant \frac{C_1(p,L_o)}{\theta_1^2} \equiv B(\theta_1) \tag{C.24}$$

$W(\theta)$ undergoes exactly one passage through zero at $\theta = \theta_n$ between two successive maxima of $|W^{(o)}(\theta)|$; if θ_{no} is the zero of $W^{(o)}(\theta)$, then:

$$|\theta_n - \theta_{no}| \leqslant \frac{C(p,L_o)}{\theta_1\sqrt{L}} \tag{C.25}$$

(iii) if L obeys (C.24), the time interval in a period of $W(\theta)$ where $(dW/d\theta)^2 > kL/2$ is bounded from below by k_1T_o, with T_o the period of $W^{(o)}$.

Comment C.4: According to Lemma C.4, those motions of (C.15) are certain to be oscillatory for which the energy is high enough. If we choose θ_1 so that, e.g. $L_o > 2\,B(\theta_1)$ (cf.(C.24)), we are certain to have oscillatory motions, at least over an interval of length pC_u. If the motion is not guaranteed to be oscillatory $(L \leqslant B(\theta_1))$, the argument of Lemma C.2 shows that:

$$L(\theta_1 + pC_u) \leqslant B(\theta_1)(1 + \alpha) \tag{C.26}$$

with:

$$\alpha = \text{const}\ (\theta_1^{-3} + \gamma\, \theta_1^{-1/4} + \varepsilon^{3/4}\ \theta_1^{1/2}) \qquad (C.27)$$

We show next:

Lemma C.5: If $L_o > 2B(\theta_1)$, then, if θ_1 is large enough:

$$L(\theta_1 + pC_u) < \frac{L_o}{1 + \frac{k_o pC_u}{\theta_1}} \equiv B_M(\theta_1 + pC_u) \qquad (C.28)$$

with some $k_o < 1$.

Proof: If $L > B(\theta_1)$, the motion is oscillatory and:

$$L(\theta_1) - L(\theta_1 + pC_u) > \int_{\theta_1}^{\theta_1+pC_u} \left(\frac{dW}{d\theta}\right)^2 \frac{3}{4\theta} d\theta - \int_{\theta_1}^{\theta_1+pC_u} \left| \frac{W^2}{2}\frac{dK}{d\theta} + \frac{|W^3|}{3}\frac{d\tilde{h}_1}{d\theta} + \frac{W^4}{12}\frac{d\tilde{h}_1^2}{d\theta}\right| d\theta \qquad (C.29)$$

Now, as in Lemma C.2, $I_2 < L(\theta_1)\alpha$, with α of (C.27). Further, for $\theta_1 > 3kpC_u/2$, $0 < k < 2$, (V is the potential in (C.10)):

$$I_1 > \frac{k}{2} L(\theta_1) \int_{V(W) < kL/2} \frac{3}{4\theta} d\theta > \frac{3k}{8} L(\theta_1) \int_{\theta_1}^{\theta_1 + k_1 pC_u} \frac{d\theta}{\theta} > \frac{k}{4} L(\theta_1) \frac{k_1 pC_u}{\theta_1} \qquad (C.30)$$

Thus:

$$L(\theta_1 + pC_u) < L(\theta_1)(1 + \alpha - \frac{k}{4}\frac{k_1 pC_u}{\theta_1}) < \frac{L(\theta_1)}{1 + k_o pC_u/\theta_1} \qquad (C.31)$$

for θ_1 large and ε small enough. With Comment C.4, we obtain:

$$L(\theta_1 + pC_u) < \max\left[L_o/(1 + k_o pC_u/\theta_1),\ B(\theta_1)(1 + \alpha)\right] \qquad (C.32)$$

Since, for θ_1 large, the maximum is achieved by the first term, the proof is completed.

Finally, we state:

Lemma C.6: For any $\sigma > 0$, there exists ε_o so that, if $\varepsilon < \varepsilon_o(\sigma)$, $L(\theta;\varepsilon)$ becomes and stays less than σ in the time interval $T_{oR}\,\varepsilon^{3p} < t < \varepsilon^{3p-\delta}$.

Comment C.5: This is the precise statement of $L(\theta;\varepsilon) \to 0$ as $\theta \to \infty$ (cf. Comment C.1)

Proof: We notice first that:

$$\frac{B_M(\theta_1)}{B(\theta_1)} < \frac{B_M(\theta_1 + pC_u)}{B(\theta_1 + pC_u)} \qquad (C.33)$$

Indeed, this is equivalent to:

$$1 + \frac{k_o pC_u}{\theta_1} < (1 + \frac{pC_u}{\theta_1})^2 \qquad (C.34)$$

which is true since $k_o < 1$. It follows that, for any m, the upper bound on the energy L at $\theta = \theta_1 + mpC_u$ is achieved by oscillatory motions. By Lemma C.5, the upper bound is:

$$B_M(\theta_1 + mpC_u) = \frac{L_o}{\prod_{j=1}^{m}\left[1 + \frac{k_o pC_u}{\theta_1 + (j-1)pC_u}\right]} < K \frac{L_o}{m^{k_o}} \tag{C.35}$$

By choosing ε small enough, we can allow for as many oscillations as we wish on $[T_{oR}\,\varepsilon^{3p}, \varepsilon^{3p-\delta}]$ and thus cause the upper bound B_M to become as small as desired. By Lemma C.1, if $B_M(\tilde{\theta}) < \sigma_1$, then $L <$ const$\cdot\sigma_1$ for $\tilde{\theta} < \theta < \varepsilon^{3p-\delta}$.

Appendix D: Proof of Theorem 5.2

Consider first the mapping: $\mathbb{P}'_L : (\Lambda, \Psi_o) \longrightarrow (R, \varphi)(\tau = -\varepsilon^{-\delta}) \equiv (R_\delta, \varphi_\delta)$ with (R, φ) of eqns. (5.14-15) and $\delta = 3k\sqrt{\pi}/16$ of Lemma 5.3.

Lemma D.1: R_δ, φ_δ have any number of derivatives with respect to Λ, Ψ_o, continuous with respect to ε as $\varepsilon \to 0$.

Proof: Consider the sequence of transformations (5.22), (5.26), supplemented by Comment 5.4. At every fixed θ, they imply:

$$\partial(R,\varphi)/\partial(R_5, \varphi_5) = 1 + O(h + g) \tag{D.1}$$

and:

$$\frac{\partial R}{\partial R_5} - 1 \sim \frac{\partial R}{\partial \varphi_5} \sim \frac{\partial \varphi}{\partial R_5} \sim \frac{\partial \varphi}{\partial \varphi_5} - 1 = O(h + g) \tag{D.2}$$

where $\sim$ means of the same order. It is thus sufficient to prove the statement of the Lemma for $R_5, \varphi_5(\tau = -\varepsilon^{-\delta}) \equiv R_{5\delta}, \varphi_{5\delta}$ as functions of Λ_{5o}, Ψ_{5o}. Now, R_5, φ_5 obey (5.23),(5.29). The right hand sides of (5.23), (5.29) are trigonometric polynomials of z, eqn. (5.21), with coefficients various powers of R (or R_5 in eqn. (5.29)). Taking derivatives on both sides of (5.23), (5.29) with respect to Λ_{5o}, we obtain a pair of linear differential equations for $\partial R_5/\partial\Lambda_{5o}$, $\partial\varphi_5/\partial\Lambda_{5o}$, on the interval $[\theta_o, \theta(\tau = -\varepsilon^{-\delta})]$. It is convenient to write:

$$\bar{\varphi}_5(\theta) = \varphi_5(\theta) + \frac{7}{24}\Lambda_{5o}^2 \int_{\theta_o}^{\theta} h^2 d\theta' \tag{D.3}$$

so that the equations for $\rho_\Lambda = \partial R/\partial\Lambda_{5o} - 1$, $\bar{\varphi}_\Lambda = \partial\bar{\varphi}_5/\partial\Lambda_{5o}$ read:

$$\frac{d}{d\theta}\rho_\Lambda = a_R(1 + \rho_\Lambda) + a_\phi\left(\bar{\varphi}_\Lambda - \frac{7}{12}\Lambda_{5o}\int_{\theta_o}^{\theta} h^2 d\theta'\right) \tag{D.4}$$

$$\frac{d}{d\theta}\bar{\varphi}_\Lambda = -\frac{7}{12}(R_5 - \Lambda_{5o})h^2 - \frac{7}{12}\rho_\Lambda R_5 h^2 + b_R(1+\rho_\Lambda) + b_\phi\left(\bar{\varphi}_\Lambda - \frac{7}{12}\Lambda_{5o}\int_{\theta_o}^{\theta} h^2 d\theta'\right) \tag{D.5}$$

where a_R, a_ϕ, b_R, b_ϕ are functions of θ (also through the solutions $R_5(\theta)$, $\varphi_5(\theta)$ of (5.23), (5.29)), with the order of magnitude shown under the $O(\cdot)$ sign in (5.23), (5.29). Integrating from θ_o to θ in (D.4),(D.5), using the initial conditions $\rho_\Lambda(\theta_o) = \bar{\varphi}_\Lambda(\theta_o) = 0$ and adding, we obtain:

$$|\rho_\Lambda(\theta)| + |\bar{\varphi}_\Lambda(\theta)| \leq s_f(\theta) + \int_{\theta_o}^{\theta} [S(\theta') + \frac{7}{12} M_5 h^2(\theta')][\,|\rho_\Lambda(\theta')| + |\bar{\varphi}_\Lambda(\theta')|\,]\, d\theta' \qquad (D.6)$$

with:

$$S(\theta') = |a_R| + |a_\phi| + |b_R| + |b_\phi| \qquad (D.7)$$

$$s_f(\theta) = \int_{\theta_o}^{\theta} \Big[(|a_R| + |b_R|)(\theta') + \frac{7}{12}\Lambda_{5o}\Big(|a_\phi(\theta')| + |b_\phi(\theta')|\Big)\int_{\theta_o}^{\theta} h^2 d\theta'' + \frac{7}{12}(R_5 - \Lambda_{5o})h^2(\theta')\Big] d\theta' \qquad (D.8)$$

and M_5 is the bound on R_5 corresponding to (5.13). Now, if $\theta_\varepsilon \equiv \theta(\tau = -\varepsilon^{-\delta})$, then, for ε small enough:

$$\int_{\theta_o}^{\theta} \Big(|S(\theta')| + \frac{7}{12} M_5 h^2(\theta')\Big)\, d\theta' < \frac{C}{(\ln \frac{1}{\varepsilon})^{2/3}} \qquad (D.9)$$

where we have used (5.52) and $0 < \beta < \beta_M$. Further, using (5.24),(5.23):

$$|s_f(\theta)| < C(\theta^{-q} + \varepsilon^{1/8}) \qquad (D.10)$$

with $q > 0$. It follows from Gronwall's inequality that (D.6) implies:

$$|\rho_\Lambda(\theta)|\,,\ |\bar{\varphi}_\Lambda(\theta)| < C_1(\theta^{-q} + \varepsilon^{q/2}) \qquad (D.11)$$

The same majorizations hold for the derivatives $\partial R_5/\partial\varphi_{5o} \equiv \rho_\phi$, $\partial\varphi_5/\partial\varphi_{5o} \equiv \varphi_\varphi \equiv \bar{\varphi}_\varphi$.

Consider next the second derivatives, e.g. with respect to Λ. Differentiating (D.4),(D.5) with respect to Λ, one obtains a set of linear equations for $\rho_{\Lambda\Lambda}, \bar{\varphi}_{\Lambda\Lambda}$; the coefficients of $\rho_{\Lambda\Lambda}, \bar{\varphi}_{\Lambda\Lambda}$ are the same as those of $\rho_\Lambda, \bar{\varphi}_\Lambda$ in (D.4),(D.5). The free term contains expressions like $(\partial a_R/\partial R_5)(1 + \rho_\Lambda)^2$, etc. The term $\partial a_R/\partial R_5$ has the same order of magnitude as the right hand side of (5.23). Using the estimates (D.11) and analogues, we conclude that the free terms are majorized like (D.10). Gronwall's inequality yields then the desired result. The same argument applies to the higher derivatives,

<u>Comment D.1</u>: The estimate:

$$\partial\varphi_5/\partial\Lambda_{5o}(-\varepsilon^{-\delta}) = -\frac{7}{12}\Lambda_{5o}^{\bullet} \int_{\theta_o}^{\theta(-\varepsilon^{-\delta})} h^2 d\theta' \qquad (D.12)$$

differs by terms of $O(\varepsilon^p)$, $p > 0$ from $\partial\tilde{\varphi}_o/\partial\Lambda$, eqn. (5.35).

<u>Lemma D.2</u>: If $\beta > \beta_m > 0$, R_δ, φ_δ have a derivative with respect to β, eqn.(5.53), continuous with respect to ε as $\varepsilon \to 0$.

<u>Proof</u>: According to (5.54), if $\beta > \beta_m$, the functions $\rho_\beta \equiv \partial R_5/\partial\beta$, $\bar{\varphi}_\beta \equiv \partial\bar{\varphi}_5/\partial\beta$

obey a linear equation with coefficients and free terms of the same order of magnitude as those of (D.4),(D.5). Inequalities similar to (D.11) may be obtained thus for $\rho_\beta, \bar{\varphi}_\beta$.

With this, we write at $\theta_\delta = \theta(\tau = -\varepsilon^{-\delta})$, using (5.12),(5.14-15),(5.40):

$$U_L(-\varepsilon^{-\delta}) = R_\delta \cos(\theta_\delta - \theta_o + \varphi_\delta)\, \varepsilon^{kt(\theta_\delta)+\delta/6} \quad ; \quad U_L'(-\varepsilon^{-\delta}) = -\sqrt{3}\, R \sin(\theta_\delta - \theta_o + \varphi_\delta) \varepsilon^{kt(\theta_\delta)-\delta/6} \tag{D.13}$$

With the help of the two solutions V_1, V_2 of (5.42), we obtain the coefficients A,B in (5.45) as:

$$A(R_\delta, \varphi_\delta) = \sqrt{3}\, R_\delta \cos(\theta_o - \varphi_\delta) \quad , \quad B(R_\delta, \varphi_\delta) = R_\delta \sqrt{3} \sin(\theta_o - \varphi_\delta) \tag{D.14}$$

Clearly, A,B have derivatives of any order with respect to R_δ, φ_δ, but these are not continuous as $\varepsilon \to 0$, since $\theta_o \sim \varepsilon^{-1/2}$. Taking derivatives with respect to R_δ, φ_δ in the integral equation (5.45), we conclude, using the boundedness of U_L of Lemma 5.5 and Gronwall's Lemma that, for ε small enough, all derivatives of U_L, U_L' exist and stay bounded as $\varepsilon \to 0$.

With Lemmas D.1,D.2, the proof of Theorem 5.2 is completed.

Comment D.2: The derivatives $\partial U_{Lo}/\partial R_\delta$, etc differ by quantities of $O(\varepsilon^s)$, $s > 0$ from the estimates:

$$\partial U_{Lo}/\partial R_\delta \sim \partial A/\partial R_\delta \cdot V_1 + \partial B/\partial R_\delta \cdot V_2, \quad \text{etc.} \tag{D.15}$$

Appendix E: Proof of Lemma 6.2

Several parts of the proof are similar to Appendix D. The change $h(\theta) \to -k(\theta)$ is not, however, totally harmless, because h,k have different orders of magnitude. Eqns. (D.1),(D.2) are the same, provided we replace O(h+g) by O(k+g) and we move over to $R_5(\theta)$, $\varphi_5(\theta)$. We denote $R_5(\theta_{oR}) = R_{5R}$, $\varphi_5(\theta_{oR}) = \varphi_{5R}$, $R_5(\theta_o) = R_{5o}$ and introduce:

$$\bar{\varphi}_5(\theta) = \varphi_5(\theta) + \frac{7}{24}\int_{\theta_{oR}}^{\theta} k^2 R_5^2 \, d\theta' \tag{E.1}$$

The equations for $\rho_R \equiv \partial R_5/\partial R_{5R} - 1$, $\bar{\varphi}_R = \partial \bar{\varphi}_5/\partial R_{5R}$ read (cf. (D.4),(D.5)):

$$\frac{d}{d\theta}\rho_R = a_R(1+\rho_R) + a_\phi\left(\bar{\varphi}_R - \frac{7}{12}\int_{\theta_{oR}}^{\theta} R_5(\theta')(1+\rho_R)\, k^2 d\theta'\right) \tag{E.2a}$$

$$\frac{d}{d\theta}\bar{\varphi}_R = b_R(1+\rho_R) + b_\phi\left(\bar{\varphi}_R - \frac{7}{12}\int_{\theta_{oR}}^{\theta} R_5(\theta')(1+\rho_R)\, k^2 d\theta'\right) \tag{E.2b}$$

with the notation of Appendix D;now, $|a_R|$, $|a_\phi|$, etc. are $O(k^4 + k^2 g)$. Integrating (E.2a,b) from θ_a to θ, inverting the order of integrals, using $\rho_R(\theta_a) = 0, \Psi_R(\theta_a) = 0$ and adding the resulting equations, we obtain the inequality:

$$|\rho_R(\theta)| + |\bar{\varphi}_R(\theta)| \leq S_f(\theta) + \int_{\theta_a}^{\theta} (S(\theta') + A(\theta, \theta'))(|\rho_R(\theta')| + |\bar{\varphi}_R(\theta')|)\, d\theta' \qquad \text{(E.3)}$$

with $S(\theta')$ of (D.7),

$$S_f(\theta) = \int_{\theta_a}^{\theta} \left[(|a_R| + |b_R|) + \frac{7}{12} (|a_\phi| + |b_\phi|) \int_{\theta_a}^{\theta'} k^2(\theta'') R_5(\theta'') d\theta'' \right] d\theta' < C\, \theta_a^{-1/3} \qquad \text{(E.4)}$$

and

$$A(\theta, \theta') = k^2(\theta') \int_{\theta'}^{\theta} (|a_\phi| + |b_\phi|) d\theta'' < C\, k^2(\theta')(\theta')^{-1/2} \qquad \text{(E.5)}$$

With (E.5), Gronwall's inequality applied to (E.3) gives a bound:

$$|\rho_R(\theta)| + |\bar{\varphi}_R(\theta)| < S_f(\theta) + C \int_{\theta_a}^{\theta} S_f(\theta')(S(\theta') + k^2(\theta')(\theta')^{-1/2})\, d\theta' < C\, \theta_a^{-1/3} \qquad \text{(E.6)}$$

Eqn. (E.6) implies that the matrix elements of the first column of $D\widetilde{\mathbb{P}}_{R2}$, eqn.(6.20), are bounded as $\varepsilon \to 0$. On the other hand, using (E.1):

$$\partial \varphi_5 / \partial R_{5a}(\theta) < \text{const} \int_{\theta_a}^{\theta} k^2 d\theta' \qquad \text{(E.7)}$$

which is unbounded as $\varepsilon \to 0$.

In the same manner, we treat $\rho_\phi = \partial R_5/\partial \varphi_{5a}$, $\bar{\varphi}_\varphi = \partial \bar{\varphi}/\partial \varphi_{5a} - 1$. However, the free terms are different, and $S_f^{(\varphi)}(\theta) = O(\theta_a^{-1/2})$. Instead of (E.7), we obtain $(\partial \varphi_5/\partial \varphi_{5a})(\theta) <$ const, as $\varepsilon \to 0$.

We prove now the continuity of $R_{oo}(\varepsilon), \bar{\varphi}_{oo}(\varepsilon)$ as $\varepsilon \to 0$. Essentially, the reasons for continuity are: (i) the initial conditions at $\theta(T_{oR})$ for eqns.(5.23), (5.29) (with $h \to -k$) are continuous as a function of ε; indeed, in eqn. (6.18), the dependence on ε (in Υ and γ_R) is such that it leads to changes of $O(\Upsilon)$ in u_R, $du_R/d\tau$ ($\tau = T_{oR}$), and thus in R_{5a}, φ_{5a} as we move away from $\varepsilon = 0$; (ii) over τ - intervals of $O(\Upsilon^{-b})$, $0 < b < 1$, $\varepsilon^{kt} \sim 1$ and we can compare directly the equations for $R_5(\theta;\varepsilon)$, $R_5(\theta;0)$ (and those for φ_5);(iii)the changes in $R_5(\theta;\varepsilon)$,$R_5(\theta;0)$ for θ intervals larger than Υ^{-b} are vanishing as $\varepsilon \to 0$. We need to make (ii) precise and fix b. The differences $\Delta R(\theta;\varepsilon) = R(\theta;\varepsilon) - R(\theta;0)$ obey:

$$\frac{d}{d\theta} \Delta R_5 = \tilde{a}_{1R} \Delta R_5 + \tilde{a}_{1\phi} \Delta \phi_5 + O(\Delta S) \qquad \text{(E.8a)}$$

$$\frac{d}{d\theta} \Delta \phi_5 = \tilde{b}_{1R} \Delta R_5 + \tilde{b}_{1\phi} \Delta \phi_5 - \frac{7}{24} \Delta(k^2 R^2) + O(\Delta S) \qquad \text{(E.8b)}$$

where $\Delta S = \Delta(k^4 + k^2 g + |dg/d\theta|)$, the difference between the values of the brackets at ε and $\varepsilon = 0$; if $\tau < \Upsilon^{-b}, 0 < b < 1$, $h(\varepsilon=0) \neq 0$; the terms $\tilde{a}_{1R}$, etc. depend on both $R(\theta;\varepsilon), R(\theta;0)$, but are bounded in the same manner as the a_R, a_ϕ, etc., of (E.2a,b), because M, eqn. (5.13), is independent of ε. The term ΔS is estimated as follows: (i) the dependence on ε is present in the term ε^{kt} of $k(\theta)$ and in the functions $\eta_R(\theta;\varepsilon)$ of both $k(\theta)$ and $g(\theta)$; (ii) it is true that $|\eta_R(\varepsilon;\tau) - \eta_R(0;\tau)| < C\,\varepsilon^{3/4}\,\tau^{7/3}$, as follows from the inner expansion (3.4); (iii) it follows that the change in k is $O(k\Delta\eta/\eta + k(1-\varepsilon^{kt})) = O(k\,\varepsilon^{3/4}\Upsilon^{-2b} + k\,\Upsilon^{1-b}) = O(k\,\Upsilon^{1-b})$ on a τ interval $\tau<\Upsilon^{-b}$; (iv) the change in g is $O(g\,\Delta\eta/\eta)$; (v) it follows that $\Delta S = O(S\cdot\Upsilon^{1-b})$. Using the inequality:

$$\Delta\varphi_5 \leq \Delta\bar{\varphi}_5 + \frac{7}{24}\int_{\theta}^{\theta}\Delta(k^2R^2)\,d\theta' \quad , \tag{E.9}$$

integrating (E.8) from θ_a to θ, inverting integrals and using the initial conditions of $O(\Upsilon)$, we conclude as before:

$$|\Delta R_5(\theta)| \quad , |\Delta\bar{\varphi}_5(\theta)| = O(\Upsilon^{1-b}) \tag{E.10}$$

for $\theta\in[\theta_a, \Upsilon^{-4b/3}]$. Finally, we add the integrals from $\Upsilon^{-4b/3}$ to $\theta_o(\varepsilon)$ (to infinity for $\varepsilon = 0$); they are $O(\Upsilon^{2b/3})$. For $b = 3/5$, we obtain, uniformly, $|\Delta R_5(\theta)|, |\Delta\bar{\varphi}_5(\theta)| = O(\Upsilon^{2/5})$, if $k = O(1)$ as $\varepsilon\to 0$. This establishes the continuity of $R_{5o}(\theta_o)$, $\bar{\varphi}_{5o}(\theta_o)$ as $\varepsilon\to 0$ and thus of $R_{oo}(\varepsilon)$, $\varphi_{oo}(\varepsilon)$, as stated in the Lemma.

Notice, if $\theta<\Upsilon^{-4b/3}$, $\Delta\varphi_5(\theta) = O(\varepsilon^c)$ as $\varepsilon\to 0$, for some $c>0$.

We turn next to the continuity in ε of the derivatives $\partial R_5(\theta)/\partial R_{5a}, \partial\bar{\varphi}(\theta)/\partial R_{5a}$, uniformly in θ, and thus of the matrix elements of $D\tilde{\mathbb{P}}_{R2}$, eqn. (6.2a). We subtract to this end the corresponding equations (E.2) written for ε and $\varepsilon=0$ and notice that, with the notation (E.8):

$$\Delta a_R = O(\Delta S + S\Delta R + S\Delta\varphi) = O(S\Delta\varphi) \tag{E.11}$$

Again, application of Gronwall's Lemma shows that, on $[\theta_a, \Upsilon^{-4b/3}]$, the changes $|\Delta\rho_R|, |\Delta\bar{\varphi}_R|$ are less than ε^c. The changes for $\theta>\Upsilon^{-4b/3}$ vanish as $\varepsilon\to 0$, which proves our statement.

We can now bound the second derivatives $\partial^2R/\partial R_a^2$, $\partial^2\bar{\varphi}/\partial R_a^2$, etc. To this end, we differentiate (E.2a,b) with respect to R_{5a} and obtain a set of ~~two~~ equations with the unknowns ρ_{RR}, $\bar{\varphi}_{RR}$. As with (E.2), we can integrate from θ_a to θ, change the order of integration, use initial conditions and obtain an inequality like (E.3) for $|\rho_{RR}| + |\bar{\varphi}_{RR}|$. Apart from this, the integral is the same as in (E.3).

The free term $S_f(\theta)$ is, however, different; it contains now contributions from ρ_R, φ_R:

$$S_f^{(2)}(\theta) = \int_{\theta_a}^{\theta}\Big[|a_{RR}|(1+\rho_R)^2 + 2|a_{R\varphi}|(1+\rho_R)\varphi_R + |a_{\varphi\varphi}|\varphi_R^2 + |a_\varphi|\int_{\theta_a}^{\theta'} k^2(\theta'')(1+\rho_R(\theta''))d\theta''\Big]d\theta' \qquad \text{(E.12)}$$

$$+ (a \longleftrightarrow b)$$

The dominant term is $a_{\varphi\varphi}\varphi_R^2$; using (E.7), we obtain:

$$S_f^{(2)}(\theta) = O(\ln \tfrac{1}{\varepsilon}) \qquad \text{(E.13)}$$

and thus

$$|\rho_{RR}(\theta)|, |\bar{\varphi}_{RR}(\theta)| = O(\ln \tfrac{1}{\varepsilon}) ; \quad \varphi_{RR} = O\Big(\int_{\theta_a}^{\theta} k^2 d\theta'\Big) \qquad \text{(E.14)}$$

A similar analysis for the other equations involving $\varphi_{\phi\phi}, \rho_{\phi R}$, etc, show that the values of the latter are bounded by constants as $\varepsilon \to 0$.

We can proceed this way for the higher derivatives; the only change, in each pair of equations, corresponding to each derivative (e.g. $\rho_{R^k\varphi^l}$, $\bar{\varphi}_{R^k\varphi^l}$) occurs in the free term. The dominant term at a given order p = k+l of the derivatives occurs in ρ_{R^p}, $\bar{\varphi}_{R^p}$ and is $\sim \int_{\theta_0}^{\theta} k^4 d\theta' \big(\int_{\theta_0}^{\theta'} k^2 d\theta''\big)^p \simeq O(\gamma^{(2-p)/3})$.

The statement concerning the derivatives with respect to β is obtained analogously.

Appendix F: Proof of Lemma 7.2

We give first some comments; the proof is then divided into a number of steps, indexed with letters.

(a) We restrict ourselves to $0<\beta<\pi$. Then, the (mod 2π) in (7.13) plays no role and we can take $-\pi<\chi<\pi$. $\mathcal{J}(\chi)$ has only two extrema, $\chi_{min} = \pi - \Sigma \ (-\pi-\Sigma)$ if $\Sigma > 0 \ (<0)$, $\chi_{max} = -\Sigma$. If $\Sigma>0\ (<0)$, $\chi_{min} > (<) \chi_{max}$.

(b) If $0<\Sigma<\pi/2$, $\beta_{2u}(\Sigma) < \beta_{2d}(\Sigma)$. Indeed, at $\beta = \pi - \Sigma$, $\mathcal{J}(\beta;\chi=\beta) \equiv \mathcal{J}(\beta;\beta) = -\pi+\Sigma < -\Sigma$. It follows that $\beta_{2u}(\Sigma) < \pi - \Sigma \equiv \chi_{min}$. Thus, χ_{min} lies outside $[-\beta_{2u}, \beta_{2u}]$ and it cannot be part of a superstable orbit of $\mathcal{J}(\beta;\chi)$, i.e. $\beta_{2d}(\Sigma) > \beta_{2u}(\Sigma)$.

If $\pi/2 < \Sigma < \pi$, $\beta_{2d}(\Sigma) < \beta_{2u}(\Sigma)$. Now, $\mathcal{J}(\beta;-\beta)\big|_{\beta=\Sigma} = \Sigma > \pi - \Sigma$. Thus, (by looking at the shape of $\mathcal{J}(\beta;-\beta)$), $-\beta_{2d}(\Sigma) > -\Sigma$, i.e. $\chi_{max} \notin [-\beta_{2d}, \beta_{2d}]$.

(c) Given $\alpha \in [-\beta, \beta]$, the equation $\overline{\mathcal{J}}(\beta;\chi) = \alpha$ cannot have more than two solutions with $|\chi| < \pi$. If $\alpha = \chi_+(\chi_-)$, a fixed point of $\overline{\mathcal{J}}$, we denote its pair by $\tilde{\chi}_+$ $(\tilde{\chi}_-)$.

(d) From (7.30), we verify $S(\overline{\mathcal{J}}) < 0$. This has the consequences (Ref.[9],p.97): (i) $S(\overline{\mathcal{J}}^p) < 0$, for all $p > 0$; (ii) $|(\overline{\mathcal{J}}^p)'|$ cannot have a strictly positive minimum; (iii) if $(\overline{\mathcal{J}}^p)'_\chi$ does not change sign on $[a,b]$ and $\overline{\mathcal{J}}^p$ has three fixed points there, the middle one is unstable and the other two stable; (iv) $\overline{\mathcal{J}}^p$ cannot have more than three fixed points on an interval $[a,b]$ with $(\overline{\mathcal{J}}^p)' > 0$ (or < 0). $(\overline{\mathcal{J}}^2 \equiv \overline{\mathcal{J}} \circ \overline{\mathcal{J}})$

(e) We describe in detail the situation at $\Sigma = -\pi/2$. Many arguments may be taken over to $\Sigma \neq -\pi/2$. At $\Sigma = -\pi/2$, for all β, $0 < \beta < \pi$, $\overline{\mathcal{J}}$ may be decomposed into two maps $\overline{\mathcal{J}}_{+,-}$ of $[-\beta, 0]$, $[0, \beta]$ into themselves. If $\beta > 1$, there are three fixed points χ_-, χ_o, $\chi_+ = -\chi_-$; χ_o is unstable. If $\beta \leq \pi/2$, it is clear geometrically that $[0, \chi_+]$, $[\chi_+, \beta]$ are contracted in turn under $\overline{\mathcal{J}}_+$ into themselves and to χ_+.

Consider next the interval $\beta_1(-\pi/2) = \pi/2 < \beta < \beta_F(-\pi/2)$; $\chi_+ > \pi/2$ and $\tilde{\chi}_+ < \pi/2$. All points of $[0, \tilde{\chi}_+]$ are mapped under $\overline{\mathcal{J}}$ eventually into $[\tilde{\chi}_+, \chi_+]$. Further, under $\overline{\mathcal{J}}$: $(\tilde{\chi}_+, \chi_+) \to (\chi_+, \beta) \to [\overline{\mathcal{J}}(\beta;\beta), \chi_+]$. Now, if $\pi/2 < \beta < \beta_{2u}(-\pi/2)$, $\overline{\mathcal{J}}(\beta;\beta) > \pi/2$. Thus, $(\overline{\mathcal{J}}^2)'_\chi > 0$ on $[\overline{\mathcal{J}}(\beta), \beta]$ and $\overline{\mathcal{J}}^2(\beta;\beta) < \beta$. $(\overline{\mathcal{J}}^2 \equiv \overline{\mathcal{J}} \circ \overline{\mathcal{J}})$ If, in addition, $\beta < \beta_F(-\pi/2)$, $(\overline{\mathcal{J}}^2)'(\chi_+) < 1$. Then, from (d)(iii),(iv) above, it follows that $\chi_+ < \overline{\mathcal{J}}^2(\chi) < \chi$, for all χ in $[\chi_+, \beta]$. Thus, $[\chi_+, \beta]$ is contracted under $\overline{\mathcal{J}}^2$ into itself and to χ_+. Therefore, the only invariant set of $\overline{\mathcal{J}}$ is $\{\chi_+\}$.

At $\beta = \beta_{2u}(-\pi/2)$, $\overline{\mathcal{J}}^2$ has at least three fixed points: $\pi/2$, χ_+, β_{2u}. From the above, $(\overline{\mathcal{J}}^2)'_\chi > 0$ on $[\overline{\mathcal{J}}(\beta_{2u}) = \pi/2, \beta_{2u}]$ and thus, from (d)(iii),(iv), there are no other fixed points of $\overline{\mathcal{J}}^2$ on $[\pi/2, \beta_{2u}]$ and $(\overline{\mathcal{J}}^2)'_\chi(\chi_+) > 1$ at $\beta = \beta_{2u}$. By the continuity in β of $(\overline{\mathcal{J}}^2)'_\chi(\chi_+)$, this is true for $\beta_F < \beta < \beta_{2u}$. Indeed, for $\overline{\mathcal{J}}(\beta;\chi)$, there is only one β_F, eqns. (7.19-20), with $0 < \beta_F < \pi$. Let then, for $\beta_F < \beta < \beta_{2u}$, χ_L, χ_R be the stable fixed points of $\overline{\mathcal{J}}^2$ in $[\overline{\mathcal{J}}(\beta), \beta]$. Since, e.g. on $(\chi_+, \chi_R]$, $\chi_R > \overline{\mathcal{J}}^2(\chi) > \chi$, $(\chi_+, \chi_R]$ is mapped into itself and to χ_R under $\overline{\mathcal{J}}^2$. The corresponding statement is true for $[\chi_L, \chi_+)$ and thus $\overline{\mathcal{J}}^2$ contracts $[0,\beta] \setminus \{\chi_+\}$ to χ_L, χ_R.

The situation for $\overline{\mathcal{J}}_-$ is symmetrical and this ends the discussion of the situation $\Sigma = -\pi/2$.

(f) We consider now the situation $-\pi/2 < \Sigma < 0$. If $\beta < \beta_s(\Sigma)$, eqns.(7.14-15), $\bar{T}$ has a single fixed point $\chi_+ > 0$. If $\beta < \beta_{1u}(\Sigma) = -\Sigma$, then $\tilde{\chi}_+ > \chi_+$. Thus, $\bar{T}(\beta;\chi) < \chi_+$ for $-\beta < \chi < \chi_+$. Since $0 < \chi_+ - \bar{T}(\beta;\chi) < \chi_+ - \chi$, it follows that $[-\beta, \chi_+]$ is contracted under $\bar{T}$ into itself and to χ_+. The same is true for $[\chi_+, \beta]$, $\beta < -\Sigma$. If $\beta > -\Sigma \equiv \beta_{1u}(\Sigma)$, $\tilde{\chi}_+ < \chi_+$; if $\beta < \beta_s(\Sigma)$, $[-\beta, \tilde{\chi}_+]$ is mapped under $\bar{T}$ eventually into $[\tilde{\chi}_+, \chi_+]$; as above $[\tilde{\chi}_+, \chi_+] \to [\chi_+, \beta] \to [\bar{T}(\beta;\beta), \chi_+]$. Since $\bar{T}(\beta;\beta) > -\Sigma$, $(\bar{T}^2)'(\chi) > 0$ on $[\bar{T}(\beta;\beta), \beta]$. Thus, as for $\Sigma = -\pi/2$, if $\beta < \beta_s(\Sigma)$ and $\beta < \beta_{2u}(\Sigma)$, $\bar{T}$ has at most one fixed point χ_+ and a pair of period two.

If $\beta > \beta_s(\Sigma)$, $\bar{T}$ has three fixed points $\chi_- < 0$, $\chi_o < 0$, $\chi_+ > 0$, χ_o is unstable. For $\bar{T}\big|_{[\chi_o,\beta]} \equiv \bar{T}_+$, the discussion above applies without changes. If $\beta < \beta_{1d}(\Sigma)$, $[\chi_-, \chi_o)$ is contracted into itself and to χ_-; also, $\tilde{\chi}_- < \chi_-$; all points in $[\beta, \tilde{\chi}_-]$ approach χ_+ under $\bar{T}$ and reach eventually $\chi > \tilde{\chi}_-$. But $[\tilde{\chi}_-, \chi_-]$ is contracted under $\bar{T}$ into itself and to χ_-. If $\beta > \beta_{1d}(\Sigma)$, $\tilde{\chi}_- > \chi_-$. The same reasoning as above shows that $(\bar{T}^2)'(\chi) > 0$ on $[-\beta, \bar{T}(-\beta;\beta)]$ so that, if $\beta < \min[\pi, \beta_{2d}(\Sigma)]$, the invariant set of $\bar{T}_- = \bar{T}\big|_{[-\beta,\chi_o]}$ consists of χ_- and a pair of points of period two (at most).

(g) If $-\pi < \Sigma < -\pi/2$, the situation is symmetrical to that for $-\pi/2 < \Sigma < 0$, in that the roles of $\bar{T}_+$, $\bar{T}_-$ are reversed; $\chi_o > 0$ and if $\beta < \beta_s(\Sigma)$, the only fixed point is χ_-.

(h) If $0 < \Sigma < \pi/2$, there is only one fixed point $\chi_+ > 0$, $\tilde{\chi}_+ < 0 < \chi_+$. The points on $[-\beta, \chi_+]$ reach eventually $[\tilde{\chi}_+, \chi_+]$. Under $\bar{T}$, $[\tilde{\chi}_+, \chi_+] \to [\chi_+, \beta] \to [\bar{T}(\beta;\beta), \chi_+]$. Since $\beta_{2u}(\Sigma) < \pi - \Sigma$ (see (b)) and $\bar{T}(\beta;\beta) > -\Sigma$ if $\beta < \beta_{2u}(\Sigma)$, it follows that $(\bar{T}^2)'_\chi > 0$ on $[\bar{T}(\beta;\beta), \beta]$. Thus, the invariant set of $\bar{T}$ contains χ_+ and possibly one pair of period two at most.

(i) If $\pi/2 < \Sigma < \pi$, the situation is symmetrical : there is only one fixed point $\chi_- < 0$, $\tilde{\chi}_- > \chi_-$ and $(\bar{T}^2)'_\chi > 0$ on $[-\beta, \bar{T}(\beta;-\beta)]$, as a consequence of $\beta_{2d}(\Sigma) < \Sigma$ (see (b)).

This ends the proof of Lemma 7.2.

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